DYNAMICS OF LOW DIMENSIONAL ORTHOGONALITY PRESERVING CUBIC STOCHASTIC OPERATORS

BY

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ABSTRACT

Cubic stochastic operator (CSO) was first introduced in 2004 by Rozikov and Khamraev. Since then, few studies had been done to study the dynamics of trajectory of some classes of CSOs. In this thesis, we consider the cubic stochastic operator (CSO) defined on 1 and 2-dimensional simplex. We provide a full description of orthogonal preserving (OP) cubic stochastic operators on the 1 and 2-dimensional simplex. We provide full description of the fixed points subject to two different parameters for the Volterra OP CSO on both simplex. In the last part of each case we described the behaviour of the fixed points. A concrete example of a non-ergodic orthogonal preserving (OP) Volterra cubic stochastic operator is given.

خلاصة البحث

تم تقديم مشغلات ستوكاستيک (CSO) لأول مرة بواسطة روزيكوف و خمريف في عام ٢٠٠٤. ومنذئذٍ تم تقديم بعض الأعمال لدراسة ديناميكيات المسار من الدرجة نفسها. في هذه الأطروحة تم النظر إلى مشغلات ستوكاستيک لتحديد بعديها الأول والثاني على المستوى البسيط. الأطروحة تعطي وصفا كاملا للمتعامد الحافظ (OP) لمشغلات ستوكاستيک على بعديها الأول والثاني على المستوى البسيط. تعطي الأطروحة وصفا كاملا لمحمل النقاط الثابتة التي تعتمد على المعلمين مختلفين (OP-CSO) لفولتيرا على كلا البعدين. في الجزء الأخير لكل حالة تصف الأطروحة سلوك النقطات الثابتة. تعطى الأطروحة الأمثلة العملية لكل من التعامد غير المحافظ ومشغلات ستوكاستيک لفولتيرا.

APPROVAL PAGE

I certify that I have supervised and read this study and that in my opinion, it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science (Computational and Theoretical Sciences).

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DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

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LIST OF SYMBOLS

$\perp$	Orthogonal
α ,β	Parameter
$\Gamma_1, \Gamma_2, \Gamma_3$	Edges of 2-dimensional simplex
∂S^2	Boundary of 2-dimensional simplex
$\in$	Element of
λ	Eigenvalues
$\mathbf{e}_i$	Standard basis / Vertices of a simplex
$Fix(V)$	The set of fixed points of an operator V
$\mathrm{int}(S^2)$	Interior of 2-dimensional simplex
J	Jacobian
$\mathbb{N}$	The set of natural numbers
$Per_k(V)$	The set of periodic points with period k of an operator V
$P_{ijk,l}$	Parameters / Coefficient
$\mathbb{R}$	The set of real numbers
S^{m-1}	$(m-1)$-dimensional simplex
S^1	1-dimensional simplex
S^2	2-dimensional simplex
U, V	Stochastic Operator

LIST OF ABBREVIATIONS

CSO	Cubic stochastic operator
OP	Orthogonal preserving
QSO	Quadratic stochastic operator

CHAPTER ONE

INTRODUCTION

1.1 LITERATURE REVIEW

The study of non-linear operator could be traced back to Bernstein's work on theory of heredity where the simplest form of non-linear operator was introduced in the form of Quadratic Stochastic Operator (QSO) (Bernstein, 1942). The quadratic operator first arose in the problem of describing the evolution of biological population and later becomes the primary source for investigation of the dynamical properties of population genetics (Hofbauer & Sigmund, 1998; Mukhamedov & Saburov, 2012). Besides biology, the QSO is also considered as an important source of analysis for the study of dynamical system and modelling in various fields such as physics (Plank, 1995), economics and finance. While investigating the QSO in general is very complicated even for lower dimension, many classes of the QSO was introduced and their asymptotic behaviour are studied, this includes classes of QSOs such as b-bistochastic QSO (Mukhamedov & Embong, 2015), dissipative QSO (Shahidi, 2008), $\xi^{(a)}$-QSO (Mukhamedov, Qaralleh, & Bt Wan Rozali, 2014), F-QSO (Rozikov & Zhamilov, 2008) and Orthogonal Preserving (OP) QSO (Mukhamedov & Taha, 2016), to name a few. However, even all these classes of QSOs together are not able to represent the QSO as a whole.

The dynamics of a QSO is extensively studied due to its applicability in describing biological system (Bernstein, 1942; Hofbauer & Sigmund, 1998). In 2013, a QSO describing the transmission of ABO and Rh blood groups was constructed by Ganikhodjaev, Saburov and Jamilov, and its dynamic was later studied numerically for the case of ABO blood groups of Malaysian people (Saburov & Arshat, 2017). The evolution of the blood groups was shown to have a unique stable equilibrium.

The Volterra QSO, in particular was used as a discrete analogy of Lotka-Volterra (LV) type model which is the simplest model of predator-prey interactions (Lyubich, 1992; Mukhamedov & Saburov, 2012). The LV type model has been studied extensively due to its ability to describe numerous natural occurrences.

Since many of its application take the quadratic form of the model (Bernstein, 1942; Hofbauer & Sigmund, 1998; Takeuchi, 1996), it led to the discretisation of the model into Volterra QSO in order to study its dynamical system.

The study on dynamics of a Volterra QSO could be dated back to 1960 when Ulam conjectured that the ergodic theorem holds for any Volterra QSO (Ulam, 1960). However, the conjecture was later proved to be false in general by a counter example given by Zakharevich (1978). Later, Zakharevich's counter example was generalised by Ganikhodjaev and Zanin (2004) for a Volterra QSO defined on 2 dimensional simplex, i.e. the necessary and sufficient conditions for the transformation to be non-ergodic was established. In 2015, the study was further generalised by Saburov (2015) to include any non-linear stochastic operator defined on 2 dimensional simplex. Previously, it was shown that a Volterra QSO is non-ergodic if there exist a Hamiltonian cycle (Ganikhodjaev, Jamilov, & Mukhitdinov, 2012), and non-ergodicity of a Volterra QSO is correlated to paper-rock-scissors zero-sum game as the zero-sum game generated by the Volterra QSO contains a Hamiltonian cycle (Ganikhodjaev, Ganikhodjaev, & Jamilov, 2015). One way to interpret a non-ergodic transformation is if its trajectory in the long run is chaotic.

Beside Volterra QSO, the notion of orthogonal preserving was introduced, and Orthogonal Preserving QSOs defined on 2-dimensional simplex were found to be permutation of Volterra QSO (Mukhamedov & Taha, 2016). Volterra operator such as Volterra QSO represents the discrete time evolution of number of species in population assuming Mendelian inheritance. The study on orthogonal preserving QSOs was later extended to include infinite dimensional case (Akın & Mukhamedov, 2015; Mukhamedov, Embong, & Pah, 2017).

Due to the absence of general theory on its dynamical system, the behaviour of classes of non-Volterra non-linear operator, including but not limited to, strictly non-Volterra QSO and quasi-strictly non-Volterra QSO was shown to be more complicated than its Volterra counterpart. In fact, some classes of QSO introduced earlier such as b-bistochastic QSO and F-QSO are non-Volterra as well (Mukhamedov & Embong, 2015; Rozikov & Zhamilov, 2008).

Study by Jamilov and Rozikov showed that strictly non-Volterra QSO has cyclic trajectory (Zhamilov & Rozikov, 2009), while later another class of non-Volterra QSO, quasi-strictly non-Volterra was shown to have a unique or two fixed points depending on its parameter (Hardin & Rozikov, 2018).

One could refer Ganikhodjaev, Mukhamedov and Rozikov (2011), and Mukhamedov and Ganikhodjaev (2015) for recent results and open problems on QSO. Although QSOs are still being extensively studied, it is natural for another class of non-linear operator such as Cubic Stochastic Operator (CSO) to be studied. The CSO which was first introduced by Rozikov and Khamraev (2004) and models the time evolution of a set of interacting components by considering the interaction between three components instead of two components is a more complex model than the QSO. The notion of CSO and classes of CSOs were introduced in several studies which, as how QSOs are; could be classified as either Volterra (Jamilov, Khamraev, & Ladra, 2018;

Rozikov & Khamraev, 2004) and non-Volterra type operator (Davronov, Jamilov, & Ladra, 2015; Jamilov et al., 2018; Rozikov & Khamraev, 2004).

Unlike QSO, there is not many studies being done on non-Volterra CSO up to recently. One such operator being studied were Conditional Cubic Stochastic Operator (CCSO), it was proven that any CCSO has a unique fixed point, and it is a regular transformation (Davronov et al., 2015). Note that regularity implies ergodicity.

Moreover, orthogonal preserving cubic stochastic operator (OP CSO) was studied and shown to be equivalent to the surjectivity of a CSO (Mukhamedov, Embong, & Rosli, 2017). The construction of OP CSOs on 2-dimensional simplex was shown in the same study to find the form of the OP CSOs. However, the dynamics of Volterra and non-Volterra OP CSO even on lower dimensional simplex are not yet fully established.

Besides, a few studies had been done on Volterra CSO. The dynamics of a special class of Volterra CSO defined on finite dimensional simplex were studied by Rozikov and Khamraev (2004).

In 2018, Jamilov, Khamraev and Ladra generalized the Volterra CSO above by introducing a class of Volterra CSO with a parameter $\theta \in [0,1]$ in the heredity coefficients. The fixed points of the Volterra operator was fully described and the asymptotical behaviour of the trajectory was investigated. They show that this class of Volterra CSO is regular as well as ergodic transformation.

However, even the Volterra operator studied by Jamlov, Khamraev and Ladra (2018) does not represent the whole Volterra CSO, the dynamics of a general Volterra CSO is still an open problem.

1.2 STATEMENT OF THE PROBLEM

It was shown that surjectivity of a CSO is equivalent to orthogonal preservability of a CSO (Mukhamedov, Fadillah Embong, et al., 2017) and OP CSOs defined on 2-dimensional simplex are permutation of Volterra CSO. However, the dynamics of OP CSOs are not fully studied yet. Basically, the study of the asymptotic behaviour of OP CSOs would involve studying both Volterra and non-Volterra CSOs. Since the two classes of Volterra CSO had been studied (Jamilov et al., 2018; Rozikov & Khamraev, 2004), we contribute to generalising the study on a CSO by studying the operators with different set of parameters or settings, but restrict it to 1-dimensional simplex for both Volterra and non-Volterra CSOs, and 2-dimensional simplex for Volterra CSO.

1.3 RESEARCH OBJECTIVES

The study aimed to achieve the following objectives:

i. to construct Orthogonal Preserving CSO defined on 1-dimensional and 2-dimensional simplex;

ii. to find and investigate the fixed points' stability of Volterra orthogonal preserving CSO defined on 1-dimensional simplex;

iii. to find and investigate the fixed points' stability of non-Volterra orthogonal preserving CSO defined on 1-dimensional simplex with condition $\alpha + \beta = 1$;

iv. to find and investigate the fixed points' stability of a Volterra orthogonal preserving CSO defined on 2-dimensional simplex.

v. to investigate the conditions of parameters of a Volterra CSO defined on 2-dimensional simplex such that the CSO is non-ergodic.

1.4 LIMITATIONS OF THE STUDY

In order to study the dynamics of a more general CSOs with more parameters compared to previous study (Jamilov et al., 2018; Rozikov & Khamraev, 2004), we couldn't generalise our result for any finite dimensional simplexes. Hence, this study needs to limit the investigation to only 1 and 2-dimensional simplexes. In our research, the CSO considered in 1-dimensional simplex is the most general. However, for 2-dimensional simplex; while it would be more general if there are 3 parameters, we only consider the case for two parameters due to difficulty.

1.5 OVERVIEW

In this work, we construct orthogonal preserving CSOs defined on 1 and 2-dimensional simplex. We also find the fixed points of some of the orthogonal preserving CSOs constrained to two parameters. Moreover, we investigate the stability of the fixed points of the orthogonal preserving CSOs. This thesis contains 6 chapters organised as follows:

In chapter one, we do literature review on non-linear stochastic operator mainly the QSO and CSO, dynamical system, and ergodicity, and we provide outline of this thesis' objectives.

In chapter two, we recall notations, definitions, previous known results and derivation that we need throughout this thesis.

In chapter three, we have the main result for orthogonal preserving CSO defined on 1-dimensional simplex, and it is divided into two sections. In the first section, we find the fixed points and investigate its stability for the Volterra case with arbitrary parameters α and β in unit square. In the second section, we find the fixed points and investigate its stability for the non-Volterra case with condition $\alpha+\beta=1$.

In chapter four, we have the main result for orthogonal preserving CSO defined on 2-dimensional simplex. In this chapter, we find the fixed points of the Volterra orthogonal preserving CSO with arbitrary parameters α and β in unit square, and we investigate their stability. We divide the chapter into two sections which are $\beta = 1 - \alpha$ and $\beta \neq 1 - \alpha$ to compare the results with previous studies.

In chapter five, we provide a detail description of a class of non-ergodic Volterra CSO defined on 2-dimensional simplex together with the conditions of its parameters under which the Volterra CSO fail the ergodic test.

In the final chapter, we conclude the thesis.

CHAPTER TWO

PRELIMINARIES

In this chapter, we will introduce the foundation of this study including definitions and previous recorded results, mainly on Cubic Stochastic Operator, orthogonal preservability, hyperbolicity of fixed point, Volterra and non-Volterra operator. Later, we will go through the construction of orthogonal preserving Cubic Stochastic Operator.

2.1 CUBIC STOCHASTIC OPERATOR

Let $E = \{1, 2, \dots, m\}$ be a collection of m number of elements and $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)})$ be the initial probability distribution of the elements. Here $\mathbf{x}^{(0)}$ is an element of the $(m-1)$-dimensional simplex,

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m \quad : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1 \right\}, \tag{2.1}$$

which is the set of probability distributions on E.

We denote $p_{ijk,l}$ as the conditional probability of having l^{th} element from the interaction between i^{th}, j^{th} and k^{th} elements. Then, the probability distribution $\mathbf{x}' = (x_1', x_2', \dots, x_m') \in S^{m-1}$ describing the relative frequencies of elements in the next state is given by

$$x_l' = \sum_{i,j,k=1}^{n} p_{ijk,l} x_i^{(0)} x_j^{(0)} x_k^{(0)}, \qquad l \in E\ . \tag{2.2}$$

The association $x^{(0)} \to x'$ defines a mapping V called evolution operator. The collection of elements evolve from an arbitrary state $x^{(0)}$ to the next state $x' = V\left(x^{(0)}\right)$, then to state $x'' = V\left(x'\right) = V\left(V\left(x^{(0)}\right)\right) = V^2\left(x^{(0)}\right)$, and so on. In other word, the operator describes the distribution of the following batch of elements if the distribution of current batch of elements is given.

A cubic stochastic operator is a self-mapping $V : S^{m-1} \to S^{m-1}$ of a simplex defined by:

$$V(\mathbf{x})_l = x'_l = \sum_{i,j,k=1}^{m} p_{ijk,l} x_i x_j x_k, \quad l \in E, \tag{2.3}$$

where $p_{ijk,l}$ satisfies

$$p_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} p_{ijk,l} = 1, \quad \forall i, j, k, l \in E, \tag{2.4}$$

and $P_{ijk,l}$ does not change for any permutation of i, j and k. Note that, for a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\mathbf{x}^{(n)},\ n = 0,1,2,\ldots$ of the initial point $\mathbf{x}^{(0)}$ under the action of operator (2.3) is defined by $\mathbf{x}^{(n+1)} = V\left(\mathbf{x}^{(n)}\right)$ where $n = 0,1,2,\ldots$. The operator (2.3) is called Volterra if for $l \notin \{i, j, k\}$, we have $p_{ijk,l} = 0,\ \forall i, j, k \in E$.

2.2 ORTHOGONAL PRESERVABILITY

We define support of $\mathbf{x} \in S^{m-1}$ as

$$Supp(\mathbf{x}) = \{i \in E : x_i \neq 0\},$$

and null of $\mathbf{x} \in S^{m-1}$ as

$$Null(\mathbf{x}) = \{i \in E : x_i = 0\}.$$

Note that $Supp(\mathbf{x}) \cup Null(\mathbf{x}) = E$. Thus we define orthogonality as follows:

Definition 2.1. Let $\mathbf{x},\mathbf{y}\in S^{m-1}$, then $\mathbf{x}$ is orthogonal or singular to $\mathbf{y}$ (denoted as $\mathbf{x}\perp\mathbf{y}$) if $Supp(\mathbf{x})\cap Supp(\mathbf{y})=\emptyset$.

Note that if $\mathbf{x}\perp\mathbf{y}$, then $\mathbf{x}\circ\mathbf{y}=0$ for any given $\mathbf{x},\mathbf{y}\in S^{m-1}$. Here the operation $\circ$ is referring to the usual dot product in $\mathbb{R}^m$.

Definition 2.2. (Orthogonal Preserving) A CSO given by (2.3) is called orthogonal preserving CSO (OP CSO) if for any $\mathbf{x},\mathbf{y}\in S^{(m-1)}$, $\mathbf{x}\perp\mathbf{y}$ implies $V(\mathbf{x})\perp V(\mathbf{y})$.

2.3 HYPERBOLICITY OF FIXED POINT

Let $\mathbf{x}\in S^{m-1}$, then we have the following definition:

Definition 2.3. A point $\mathbf{x}$ is called a fixed point of a CSO V if $V(\mathbf{x})=\mathbf{x}$.

Definition 2.4. A point $\mathbf{x}$ is called a periodic point of a CSO V with period k if $V^k(\mathbf{x})=\mathbf{x}$.

From both definitions we define a set of fixed points as

$$Fix(V)=\{\mathbf{x}\in S^{m-1}|V(\mathbf{x})=\mathbf{x}\},$$

and a set of periodic points as

$$Per_k(V)=\{\mathbf{x}\in S^{m-1}|V^k(\mathbf{x})=\mathbf{x}\}.$$

Note that if $k=1$, then $Per_1(V)=Fix(V)$. To show the existence of a fixed point in S^{m-1} with respect to operator (2.3), the following theorem is well known:

Theorem 2.1. (Brouwer Fixed-Point Theorem in $\mathbb{R}$) Given that set $K\subset\mathbb{R}^m$ is compact and convex, and that function $f:K\to K$ is continuous, then there exist some $c\in K$ such that $f(c)=c$.

It is evident that the segment $[0,1]\in\mathbb{R}$ is compact, then S^{m-1} is compact convex. Since operator (2.3) is a mapping $V:S^{m-1}\to S^{m-1}$, by Brouwer Fixed-Point Theorem, we always have $Fix(V)$ to be non-empty for any CSO V.

Definition 2.5. Let $V: S^{m-1} \to S^{m-1}$ be an operator defined as (2.3), then the Jacobian matrix of V at a point $\mathbf{x} = (x_1, x_2, \dots, x_m) \in S^{m-1}$ denoted by $DV(\mathbf{x})$ is given by

$$DV(\mathbf{x}) = \frac{\partial(V_1, V_2, \dots, V_m)}{\partial(x_1, x_2, \dots, x_m)} = \begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \cdots & \frac{\partial V_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial V_m}{\partial x_1} & \cdots & \frac{\partial V_m}{\partial x_m} \end{bmatrix}.$$

Let λ_n be the eigenvalues of the Jacobian matrix $DV(\mathbf{x})$ where $n \in \mathbb{N}$, then we have the following definitions:

Definition 2.6. (Devaney, 2003) A fixed point $\mathbf{x}$ is called hyperbolic if $|\lambda_n| \neq 1$ for any n.

Definition 2.7. (Devaney, 2003) Let $\mathbf{x} \in Per_k(V)$, then

i. $\mathbf{x}$ is a sink or attracting periodic point if $|\lambda_n| < 1$ for all n;

ii. $\mathbf{x}$ is a source or repelling periodic point if $|\lambda_n| > 1$ for all n;

iii. $\mathbf{x}$ is a saddle-node if otherwise.

2.4 VOLTERRA AND NON-VOLTERRA CSO

Definition 2.8. The operator (2.3) is called Volterra CSO if and only if the parameter $p_{ijk,l}$ of the operator satisfies

$$p_{ijk,l} = 0 \quad if \quad l \notin \{i, j, k\}.$$

Volterra CSO corresponding to (2.3) would have the following form:

$$V(\mathbf{x})_l = x_l \left(x_l^2 + 3x_l \sum_{i \in E \setminus \{l\}} p_{ill,l} x_i + 3 \sum_{i \in E \setminus \{l\}} p_{iil,l} x_i^2 + 6 \sum_{\substack{i,j \in E \setminus \{l\} \\ i<j}} p_{ijl,l} x_i x_j \right), \tag{2.5}$$

or

$$V(\mathbf{x})_l = x_l \left(x_l^2 + x_l \sum_{i \in E \setminus \{l\}} a_{il} x_i + \sum_{i,j \in E \setminus \{l\}} b_{ijl} x_i x_j \right), \quad l \in E, \tag{2.6}$$

where $a_{il} = 3p_{ill,l}$ and $b_{ijl} = 3p_{ijl,l}$ are coefficients depending on $p_{ijk,l}$ (Jamilov et al., 2018).

In 2004, the following Volterra CSO was studied by Rozikov and Khamraev:

$$x_l' = x_l\left(x_l^2 + 3x_l\sum_{i\in E\setminus\{l\}} x_i + 2\sum_{\substack{i,j\in E\setminus\{l\}\\ i<j}} x_i x_j\right), \quad l\in E, \tag{2.7}$$

which is then further generalised by Jamilov, Khamraev and Ladra (2018) to

$$x_l' = x_l\left(x_l^2 + 3\theta x_l\sum_{i\in E\setminus\{l\}} x_i + 3(1-\theta)\sum_{i\in E\setminus\{l\}} x_i^2 + 2\sum_{\substack{i,j\in E\setminus\{l\}\\ i<j}} x_i x_j\right), \quad l\in E, \tag{2.8}$$

where $\theta \in [0,1] \setminus \left\{\frac{2}{3}\right\}$. Note that if $\theta = 1$, then the operator (2.8) is equal to the operator (2.7).

The dynamics of operator (2.8) were studied and based on the following definition:

Definition 2.9. (Jamilov et al., 2018) We define the set ∂S^{m-1}, M_{uv}, Γ_α, $\text{int}\left(S^{m-1}\right)$, $\mathbf{e}_i$ and C as the following:

i. the set $\partial S^{m-1} = \{\mathbf{x}\in S^{m-1} : x_i = 0 \text{ for at least one } i\in E\}$ is the boundary of the simplex S^{m-1};

ii. the set $M_{uv} = \{\mathbf{x}\in S^{m-1} : x_u = x_v\}$, $u,v\in E$ is the median of the simplex;

iii. the set $\Gamma_\alpha = \{\mathbf{x}\in S^{m-1} : x_i = 0,\ i\in\alpha\subset E\}$ is the face of the simplex;

iv. the set $\text{int}\left(S^{m-1}\right) = \{\mathbf{x}\in S^{m-1} : x_1x_2\cdots x_m > 0\}$ is the interior of the simplex;

v. the set $\mathbf{e}_i = (\delta_{1i},\delta_{2i},\dots,\delta_{mi})\in S^{m-1}$, $i = 1,\dots,m$ is the vertex of the simplex;