# ON $\xi^{-}$QUADRATIC STOCHASTIC OPERATORS AND RELATED ALGEBRAS

BY

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# ABSTRACT

In this thesis, we start to study a class of quadratic stochastic operators called $\xi^{(s)}$-QSO. We first classify them into 20 non-conjugate classes. Moreover, we investigate the dynamics of four classes of $\xi^{(s)}$-QSO. Furthermore, we study another class of quadratic stochastic operator called $\xi^{(a)}$-QSO. We also classify $\xi^{(a)}$-QSO into two non-conjugate classes. Further, we investigate the dynamics of these classes. After that, we move to study the existence of associativity and derivations of genetic algebras generated by the four classes of $\xi^{(s)}$. Moreover, we figure out the connection between genetic and evolution algebras. Thereafter, we reduce the study of arbitrary evolution algebra of permutations into two special evolution algebras. Furthermore, we establish some properties of three-dimensional evolution algebras whose each basis element has infinite period. At end, we classify three dimension nilpotent and solvable evolution algebras.

# خلاصة البحث

في هذه الاطروحة نبدا في دراسة مجموعه من المؤترات التربيعية العشوائية $\xi^{(s)}-QSO$ في البداية نصنف هذه المؤثرات الى 20 مجموعة غير متاطبقة ونقوم بمناقشة السلوك الحركي لاربعة مجموعات منها بعد ذلك نذهب لدراسة نوع اخر من المؤثرات التربيعية العشوائية والتى تدعى $\xi^{(s)}-QSO$ ونقوم بتصنيفها الى مجموعتين غير متطابقتين وندرس السلوك الحركي للمجموعتين بعد ذالك نقوم بدراسة الخصائص التجمعية والمشتقات للجبر الوراثي المتولد من الاربع مجموعات التي تم دراستها وزيادة على ذلك نقوم بايجاد العلاقة بين الجبر الوراثي والجبر التطوري ثم نقوم بتقليص دراسة الجبر التطوري التبديلي الى مثالين بعد ذالك نقوم بايجاد بعض الخصائص للجبر التطوري في البعد الثالث الذي يملك اساساتة تناوب منتهي في النهاية نقوم بتصنيف الجبر التطوري القابل للحل في البعد الثالث

# APPROVAL PAGE

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# DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions

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Affirmed by Izzat Salem Qaralleh.

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Signature Date

***Dedicated to***

*My parents and my brothers and sisters*

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# LIST OF COMMON SYMBOLS / ABBREVIATIONS

| | |
|---|---|
| $S^{m-1}$ | $m-1$-dimensional simplex |
| R | The set of real numbers |
| $C$ | The set of complex numbers |
| $P_{ij,k}$ | Coefficient |
| $\Gamma$ | Boundary of simplex |
| $J$ | Jacobian |
| $\omega_V\left(x^{(0)}\right)$ | Set of limiting points of the trajectory |
| **G** | Genetic algebras |
| E | Evolution algebras |
| V | Operator |
| Fix(V) | Fixed points of operator V |

# CHAPTER ONE

# INTRODUCTION

## 1.1 LITERATURE REVIEW

The history of quadratic stochastic operators can be traced to Bernstein's work (Bernstein, 1924). The quadratic stochastic operators were considered an important source of analysis for the study of dynamic properties and models in various fields such as biology (Bernstein, 1924; Hofbaueret al., 1987; Hofbauer and Sigmund, 1988; LI et al., 2006; Lotka, 1920; Lyubich et al., 1992; Volterra, 1926), physics (Plank and Losert, 1995; Udwadia and Raju, 1998), economics and mathematics (Hofbauer and Sigmund, 1988; Kesten, 1970; Lyubich et al., 1992; Ulam, 1964). It is known that the theory of Markov processes has many applications to various branches of physics and mathematics. at the same time, this theory is not applicable to describe many physical and biological systems. One of such system is given by quadratic stochastic operators (QSO) which are related to population genetics (Bernstein, 1924). The fascinating applications of QSO to population genetics were given by Lyubich et al. (1992). As an application of a non-Mendlain inheritance, QSO has been constructed to describe transmission of ABO and Rh blood group by Ganikhodjaev et al. (2013). In Ganikhodjaev et al. (2011), it was given along self-contained exposition of the recent achievements and open problems in the theory of the QSO. The dynamics of QSO have been mentioned in Ulum (1964). The limit behaviors and argodic properties of trajectories of QSO and their application to population genetics have been investigated by Kesten (1970), Zakharevich (1978), Ganikhodzhaev (1993) and Lyubich (1971). Instead, Lotka-Volterra (LV) systems are usually used to represent the time evolution of differing species in biology (Bernstein, 1924; Volterra and Brelot, 1931). LV systems have been deeply studied by Lotka (1920) and Volterra and Brelot (1931). Moreover, LV systems have been used to describe numerous natural phenomena (Takeuchi, 1996). The most important work for an LV discrete-time system is as a recognized subject in applied mathematics (Lyubich et al., 1992). LV system was used for the first time in a biomathematical framework in Moran (1950) and improved by May and Oster (1976), which was the departure point to investigate dynamic properties and models in different areas starting from biology (Bernstein, 1924; Hofbauer et al., 1987; Hofbauer and Sigmund, 1988; LI et al., 2006; Lotka, 1920; Lyubich et al., 1992; Volterra, 1926), and from physics (Plank and Losert, 1995; Udwadia and Raju, 1998), to economics and mathematics (Hofbauer and Sigmund, 1988; Kesten, 1970; Lyubich et al., 1992; Ulam, 1964) which have been using LV systems as a source of analysis.The majority of these applications have taken the quadratic form of the LV system, which lead to the discretization of dynamic systems in order to study the computational side of such systems. This implies the study of the trajectory of discrete time of Voltera operators. In papers by Ganikhodzhaev (1993); Ganikhodzaev (1994), Ganikhodzhaev and Abdirakhmanova (2002), Ganikhodzhaev and Dzhurabaev (1998),Ganikhodzhaev and Abdirakhmanova (2002); Ganikhodzhaev and ` Eshmamatova (2006),Ganikhodzhaev and Saburov (2008), Ganikhodzhaev and Zanin (2004),Mukhamedov et al. (2005), Mukhamedov and Saburov (2009) Voltera operators and permutated Voltera operators are studied and considered discrete time. The difficulty of studying asymptotic behavior even in small dimensions forced the researchers to introduce certain classes of QSO and investigate their behavior. Namely, Volterra-QSO (Ganikhodzhaev, 1993; Ganikhodzhaev and ` Eshmamatova, 2006; Jenks, 1969; Ulam, 1964), permutated Volterra-QSO (Ganikhodzhaev and Dzhurabaev, 1998; Ganikhodzhaev and Abdirakhmanova, 2002), Quasi-Volterra-QSO

(Ganikhodzhaev, 1989), l-Volterra-QSO (Rozikov and Zada, 2009, 2010), non-Volterra-QSO (Ganikhodzhaev, 1989), strictly non-Volterra-QSO (Zhamilov and Rozikov, 2009), F-QSO (Rozikov and Zhamilov, 2008), and non Volterra operators generated by product measure (Ganikhodzhaev and Eshmamatova, 2006; Ganikhodzhaev, 2000; Rozikov and Shamsiddinov, 2009). However, all these classes together would not cover the set of all QSO. Recently, in the papers (Mukhamedov and Jamal, 2010; Mukhamedov et al., 2012) a new class of QSO was introduced. This class was called a $\xi^{(s)}$-QSO. In this work we continue the study of $\xi^{(s)}$-QSO. This class of operators depends on a partition of the coupled index set (the coupled trait set) $P_m = \{(i, j): i < j\} \subset I \times I$. In the case of two dimensional simplex ($m = 3$), the coupled index set (the coupled trait set) $P_3$ has five possible partitions. The dynamics of $\xi^{(s)}$-QSO corresponding to the point partition (the maximal partition) of $P_3$ have been investigated in Mukhamedov and Jamal (2010), Mukhamedov et al. (2012). In the present work, we describe and classify such operators generated by three other partitions. Further, we also investigate the dynamics of four classes of such operators.

It is known that each QSO generate genetic algebras (Lyubich, 1971), which is commutative and non-associative in general. The interpretation of genetic algebras is very useful. Therefore, it was necessary to study the properties of such an algebra. Some properties were given in Lyubich (1971). The existence of associative algebras with genetic realization is proved in Ganikhodjaev (2008). Generally, the associativity and derivations of genetic algebra have not been fully studied yet. Note that for any algebra $E$, the space $Der(E)$ of all its derivations is a Lie algebra with respect to the commutator multiplication. In the theory of non-associative algebras, particularly, in genetic algebras, the Lie algebra of derivations of a given algebra is one of the important tools for studying its structure. There has been much work on the subject of derivations of genetic algebras (Costa (1982),Costa (1983),Gonshor (1988)). For evolution algebras the system of equations describing the derivations are given in Tian (2008).

In Micali and Revoy (1986) it was showed that the multiplication is defined in terms of derivations, showing the significance of derivations in genetic algebras. Several genetic interpretations of derivation of genetic algebra are given in Holgate (1987). In this work we study associativity of such an algebra related to $\xi^{(s)}$-QSO. Moreover, we investigate its derivations. In Tian (2008), a new kind of algebra which is called evolution algebra has been introduced, and the question was about the related between genetic algebra and evolution algebra. In this work we will show the relation between genetic and evolution algebras in two dimensions, which leads to apply many results of evolution algebras into genetic algebras. Now, let us turn to review some work on evolution algebras. The concept of evolution algebra lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebras; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc. In Tian (2008) a foundation of the framework of the theory of evolution algebras is established and some applications of evolution algebras in the theory of stochastic processes and genetics are discussed. Recently, Rozikov and Tian (2011) studied algebraic structures evolution algebras associated with Gibbs measures defined on some graphs. In Camacho et al. (2013), Casas et al. (2011), Ladra et al. (2011), derivations, some properties of chain of evolution algebras and dibaricity of evolution algebras have been studied. In Camacho et al. (2010), Casas et al. (2013), certain algebraic properties of evolution algebras (like right nilpotency, nilpotency and solvability etc.) in terms of structure of matrix constants have been investigated. In fact, nilpotency, right nilpotency and solvability might be interpreted in a biological way as a various type of vanishing ("deaths") populations. In Casas et al. (2010), some properties of $n$-dimensional nilpotent evolution algebra have been studied. In Camacho et al. (2010), it was proven that any $n$-dimensional right nilpotent evolution algebras is nilpotent. Moreover, evolution algebra of dimension $n$ described some possible values for index of nilpotency and proved that $1+2^{n-1}$ is a maximal nilpotency index. In paper Casas et al. (2010) it was given the classification of two dimension complex evolution algebras. We should stress that classification of three dimensional complex evolution algebra is very huge and tricky, because of that we restrict ourself to classify three dimension solvable evolution algebras

**1.2 OBJECTIVES**

In this thesis, the our main objectives of the study are the following:

1. Classify $\xi^{(as)}$-QSO into non-conjugate classes. Moreover, investigate the dynamics of some classes of $\xi^{(as)}$-QSO.
2. Study the existence of associative genetic algebras generated by some class of $\xi^{(s)}$-QSO.
3. Describe the derivations of three dimension genetic algebras.
4. Reduce the study of evolution algebras of permutations into two special types of evolution algebras. Neamly, idempotents and absolute nilpotent elements of the algebra.

5. Study three-dimensional evolution algebras whose each element of evolution basis has an infinite period. and classify all three dimension nilpotent evolution algebras. In addition , classify all three dimension nilpotent evolution algebras.

**1.3 OVERVIEW**

In this work, we investigate the dynamic of QSO for the class of $\xi^{(s)}$ and $\xi^{(a)}$ and we also study genetic algebras generated by $\xi^{(s)}$-QSO. Moreover, we find the connection between genetic and evolution algebras in dimension two, which leads us to study through this work some properties of evolution algebras. This thesis contains eight chapters which are organized as follows: In chapter one, we do the literature review on quadratic stochastic operators, genetic algebras, evolution algebras and the outline of the objectives of this thesis.

In chapter two, we recall some notations and basic definitions, which are needed throughout this thesis.

Chapter three contains five sections. In section one, we classify $\xi^{(s)}$-QSO into 20 non-conjugate classes. In section two, we study dynamics of $\xi^{(s)}$-QSO taken from the class $K_1$. In section three, we investigate dynamics of $\xi^{(s)}$-QSO taken from the class $K_4$. Section four is devoted to investigate the dynamics of $\xi^{(s)}$-QSO taken from the class $K_{19}$. In the last section of this chapter, we study dynamics of $\xi^{(s)}$-QSO taken from the class $K_{25}$.

Chapter four consists of four sections. In section one, we describe 36 operators. In section two, we classify $\xi^{(a)}$-QSO into two non-conjugate classes.

Section three investigates the dynamics of class $V_1$ taken from the class $L_1$. In the last section of this chapter, we study the dynamics of $V_2$ taken from the class $L_2$.

Chapter five has seven sections. In sections 1-4, we study the associativity of genetic algebras corresponding to the operators taken from the classes $K_1, K_4, K_{17}$ and $K_{19}$ respectively. In section five, we describe the derivation of three dimension genetics algebra. Furthermore, we investigate the derivations of 3-dimensional genetic algebras related to classes $K_1, K_4, K_{17}$ and $K_{19}$. Section six is devoted to show the relation between genetic and evolution algebras. In the last section of this chapter, we show the existence of nontrivial derivations of genetic algebras.

Chapter six is organized as follows: In section one, we reduce the study of arbitrary evolution algebra of permutations into two special evolution algebras. Section two is devoted to the description of $n$-dimensional associative enveloping algebras of $n$-dimensional evolution algebras with some restrictions on $rank$ of the matrix $A$ of structural constants. Moreover, associative enveloping algebras for $2$-dimensional evolution algebras are described. In section 3 we establish some properties of three-dimensional evolution algebras whose each basis element has an infinite period.

Chapter seven has three sections. Section one describes three dimensional nilpotent evolution algebras, three dimensional nilpotent evolution algebras are classified into three classes. In section two, we describe one dimensional subalgebra of the nilpotent evolution algebras. In section three, we classify the three dimensional solvable evolution algebras. In the last chapter we conclude the thesis.

# CHAPTER TWO

# PRELIMINARIES

## 2.1 QUADRATIC STOCHASTIC OPERATOR

In this chapter, we recall all definitions, theorems, lemmas, and corollaries which are needed throughout this thesis. A quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{m-1} = \left\{ x = (x_1, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1, \ x_i \geq 0, \ i = \overline{1,m} \right\} \tag{2.1.1}$$

into itself, of the form

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1,m}, \tag{2.1.2}$$

where $V(x) = x' = (x'_1, \cdots, x'_m)$ and $\{P_{ij,k}\}$ are coefficients of heredity, which satisfy the following conditions

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1. \tag{2.1.3}$$

Thus, each quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ can be uniquely defined by a cubic matrix $\mathbf{P} = (P_{ijk})_{i,j,k=1}^{m}$ with conditions (2.1.3). We denote the sets of fixed points and $k-$periodic points of $V : S^{m-1} \to S^{m-1}$ by $Fix(V)$ and $Per_k(V)$, respectively. Due to Brouwer's fixed point theorem, for any QSO $V$ one always has that $Fix(V) \neq \varnothing.$ For a given point $x^{(0)} \in S^{m-1}$, a trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of $V$ starting from $x^{(0)}$ is defined by $x^{(n+1)} = V(x^{(n)})$. By $\omega_V\left(x^{(0)}\right)$, we denote the set of limiting points of the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$. Since $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and $S^{m-1}$ is compact, one has that $\omega_V\left(x^{(0)}\right) \neq \varnothing$. Obviously, if $\omega_V(x^{(0)})$ consists of a single point, then the trajectory converges and a limiting point is a fixed point of $V$. Recall that a Volterra-QSO is defined by (2.1.2), (2.1.3) and the additional assumption

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}. \tag{2.1.4}$$

The biological treatment of condition (1.2.4)is clear: the offspring repeats the genotype (trait) of one of its parents. One can see that a Volterra-QSO has the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \ k \in I, \tag{2.1.5}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \quad for\, i \neq k \; and \; a_{ii} = 0, i \in I. \tag{2.1.6}$$

Moreover,

$$a_{ki} = -a_{ik} \quad and \quad |a_{ki}| \leq 1.$$

This kind of operator was intensively studied in Dohtani (1992), Ganikhodzhaev (1993), Ganikhodzaev (1994), Ganikhodzhaev and ` Eshmamatova (2006), Jenks (1969). Note that this operator is a discretization of the Lotka--Volterra model (Lotka, 1920; Volterra,1926) which models an interacting competing species in the population system. Such a model has received considerable attention in the fields of biology, ecology, mathematics (see for example (Hofbauer et al., 1987; Hofbauer and Sigmund, 1988; Plank and Losert,1995; Volterra, 1926)). Rozikov and Zada (2010) introduced a notion of $\ell$-Volterra-QSO, which generalizes a notion of Volterra-QSO. Let us recall it here. In order to introduce a new class of QSO, we need some auxiliary notations. We fix $\ell \in I$ and assume that elements $P_{ij,k}$ of the matrix $(P_{ij,k})_{i,j,k=1}^{m}$ satisfy

$$P_{ij,k} = 0 \;\; if \quad k \notin \{i, j\} \quad for \quad any \;\; k \in \{1, \ldots, \ell\}, \; i, j \in I, \tag{2.1.7}$$

$$P_{i_0 j_0,k} > 0 \text{ for some } (i_0, j_0), i_0 \neq k,\ j_0 \neq k,\ k \in \{\ell+1,\ldots,m\}. \qquad (2.1.8)$$

The QSO defined by (2.1.2), (2.1.3), (2.1.7)and (2.1.8) is called $\ell$-*Volterra-QSO*.

*Remark 2.1.1 Here, we stress the following points:*

1. Note that an $\ell$-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.
2. It is known (Ganikhodzhaev1993) that there is not a periodic trajectory for Volterra-QSO. However, there are such trajectories for $\ell$-Volterra-QSO (Rozikov and Zada, 2010).

Following (Rozikov and Zada, 2010), take $k \in \{1,...,\ell\}$, then $P_{kk,i} = 0$ for $i \neq k$ and

$$1 = \sum_{i=1}^{m} P_{kk,i} = P_{kk,k} + \sum_{i=\ell+1}^{m} P_{kk,i}.$$

By using $P_{ij,k} = P_{ji,k}$ and denoting $a_{ki} = 2P_{ik,k} - 1, k \neq i$, $a_{kk} = P_{kk,k} - 1$ one then gets

$$V: \begin{cases} x_{k'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) \text{ if } k = \overline{1,\ell} \\ x_{k'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) + \sum_{\substack{i,j=1 \\ i\neq k,\, j\neq k}}^{m} P_{ij,k} x_i x_j \text{ if } k = \overline{\ell+1,m}. \end{cases} \qquad (2.1.9)$$

This is a canonical form of $\ell$-Volterra-QSO. Note that

$$a_{kk} \in [-1,0], |a_{ki}| \leq 1, a_{ki} + a_{ik} = 2(P_{ik,i} + P_{ik,k}) - 2 \leq 0,\ i,k \in I.$$

We say that an operator $V$ is *permuted* $\ell$-*Volterra-QSO*, if there is a permutation $\tau$ of the set $I$ and an $\ell$-Volterra-QSO $V_0$ such that $(V(x))_{\tau(k)} = (V_0(x))_k$ for any $k \in I$. In other words, $V$ can be represented as follows:

$$V_\tau: \begin{cases} x_{\tau(k)}' = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) \text{ if } k = \overline{1,\ell} \\ x_{\tau(k)}' = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) + \sum_{\substack{i,j=1 \\ i\neq k,\, j\neq k}}^{m} P_{ij,k} x_i x_j \text{ if } k = \overline{\ell+1,m}. \end{cases} \qquad (2.1.10)$$

We remark that if $\ell = m$ then a permuted $\ell$-Volterra-QSO becomes a permuted Volterra-QSO. Some properties of such operators were studied in Ganikhodzhaev and` Eshmamatova (2006),Ganikhodzhaev and Karimov (2000).. The dynamics of a certain class of permuted Volterra-QSO has been investigated in Mukhamedov et al. (2012). Note that Rozikov and Zada (2010), Rozikov et al. (2012) studied a class of $\ell$-Volterra-QSO. An asymptotic behavior of permuted $\ell$-Volterra-QSO has not been investigated yet. Some particular cases have been considered in Mukhamedov and Jamal (2010). In this thesis, we are going to introduce a new class of QSO which contains $\ell$-Volterra-QSO and permuted $\ell$-Volterra-QSO as a particular case. In Ganikhodjaev and Mukhitdinov(2003) a class of Quasi-Volterra operators is introduced. For such operators the condition (2.1.5) is not satisfied only for very few values of $i, j, k$. In Ganikhodzhaev (1989) it was considered the following family of QSOs $V_\lambda : S^2 \to S^2$ :

$$V_\lambda = (1-\lambda)V_0 + \lambda V_1, \quad 0 \le \lambda \le 1,$$

where

$$V_0(x) = \left(x_1^2 + 2x_1x_2, x_2^2 + 2x_2x_3, x_3^2 + 2x_1x_3\right),$$

is Volterra operator and

$$V_1(x) = \left(x_1^2 + 2x_2x_3,\ x_2^2 + 2x_1x_3,\ x_3^2 + 2x_1x_2\right),$$

is non-Volterra QSO. Note that behavior of the trajectories of $V_0$ is very irregular (Lyubich et al., 1992;Ulam, 1964; Zakharevich, 1978).

Ganikhodzhaev (2000), Ganikhodzhaev and Rozikov (2006) gave a constructive description of the matrix P. This construction depends on a probability measure $\mu$ which is given on a fixed graph $G$ and cardinality of a set of cells (configurations). Such kind of operators are defined Non-Volterra QSO generated by a product measure. In Ganikhodzhaev (2000) it was proven that the QSO constructed by the construction is Volterra if and only if $G$ is a connected graph.

Rozikov and Shamsiddinov (2009) described a class of non-Volterra QSOs using the construction of QSO for the general finite graph and probability measure $\mu$ (here $\mu$ is product of measures defined on maximal subgraphs of the graph $G$). It was shown that if $\mu$ is given by the product of the probability measures then corresponding non-Volterra operators can be studied by $N$ number (where $N$ is the number of maximal connected subgraphs) of Volterra operators defined on the maximal connected subgraphs. Consider $E_0 = E\bigcup\{0\} = \{0,1,...,m\}$. Fix a set $F \subset E$ and call this set the set of "females" and the set $M = E \setminus F$ is called the set of "males". The element $0$ will play the role of "empty-body". Coefficients $P_{ij,k}$ of the matrix $\mathbf{P}$ we define as follows

$$P_{ij,k} = \begin{cases} 1, \text{ if } & k = 0, i, j \in F \cup \{0\} \text{ or } i, j \in M \cup \{0\}; \\ 0, \text{ if } & k \neq 0, i, j \in F \cup \{0\} \text{ or } i, j \in M \cup \{0\}; \\ \geq 0, \text{ if } & i \in F, j \in M, \forall k. \end{cases} \qquad (2.1.11)$$

Biological treatment of the coefficients (2.1.11) is very clear: a "child" $k$ can be generated if its parents are taken from different classes $F$ and $M$. For a given $F \subset E$ a QSO with (2.1.11) is called a $F$-QSO. Note that $F$-QSO is non-Volterra for any $F \subset E$. In Rozikov and Zhamilov (2008) the $F$-QSOs are studied for any $F \subset E$. It was proven that such operators have unique fixed points $(1,0,...,0) \in S^m$ and all trajectories converge to this fixed point faster than any geometric progression. Recently in Zhamilov and Rozikov (2009) a new class of non-Volterra operators is introduced. These operators satisfy

$$P_{ij,k} = 0, \text{ if } k \in \{i, j\}, \forall i, j, k \in E. \qquad (2.1.12)$$

Such operator is called strictly non-Volterra QSO. For arbitrary strictly non-Volterra QSO defined on $S^2$ in Zhamilov and Rozikov (2009) it was proved that every such an operator has a unique fixed point. Also it was proven that such operators have a cyclic trajectory. This is quite different behavior from the behavior of Volterra operators, since the Volterra operators have no cyclic trajectory.

**Definition 2.1.2** *If $f : X \to X$ is a function and $f(c) = c$, then $c$ is a fixed point of $f$. Such fixed point(s) are denoted as $Fix(x)$.*

**Definition 2.1.3** *The point $x$ is called periodic point of $f$ with the period $k$ if $f^k(x) = x.$ In other words, $x$ is a periodic point of $f$ with period $k$ if $x$ is a fixed point of $f^k$. The point $x$ has prime period $k_0$ if $f_0^k(x) = x$ and $f^n(x) \neq x$ whenever $0 < k_0 < n$. That is, $x$ has prime period $k_0$ if $x$ returns back to its starting place for the first time after exactly $k_0$ iterations of $f$. Such point(s) are denoted as $Per_{k_0}(x)$.*

*in addition, the investigation of the dynamics is to investigate whether the fixed point is attracting or repelling. To begin with, we assume the mapping $f : X \to X$ is given by the following form*

$$f(x_1, x_2,..., x_m) = (f_1(x_1, x_2,..., x_m), f_2(x_1, x_2,..., x_m),..., f_3(x_1, x_2,..., x_m)).$$

*We let $A$ be an $m \times m$ matrix. An eigenvalue of $A$ is a root of the characteristics polynomial of $A$ is given by $\rho(\lambda) = det(A - \lambda E)$. We denote a Jacobian of $f$ at point $x$ by*

$$J(f(x)) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}$$