# ON GENERALIZED *P*-ADIC LOGISTIC DYNAMICAL SYSTEM

BY

WAN NUR FAIRUZ ALWANI BINTI WAN ROZALI

A thesis submitted in fulfilment of the requirement
for the degree of Master of Science (Computational And Theoretical Sciences)

Kulliyyah of Science
International Islamic University
Malaysia

JUNE 2012

# ABSTRACT

Applications of *p*-adic numbers in *p*-adic mathematical physics, quantum mechanics stimulated increasing interest in the study of *p*-adic dynamical system. One of the interesting *p*-adic dynamical system is *p*-adic logistic map. It is known such a mapping is chaotic. In the present thesis, we consider its cubic generalization, namely we study a dynamical system of the form $f(x) = ax(1 - x^2)$.Thethesis is devoted to the investigation of trajectory of the given system. We investigate the generalized logistic dynamical system with respect to parameter $a$. For the value of parameter, we consider three cases: $|a|_p < 1, |a|_p > 1$ and $|a|_p = 1$. For each case, we study the existence of the fixed points. Moreover, 2-periodic points are also studied for the case $|a|_p < 1$.Not only that, their behavior also being investigated whether such fixed points are attracting, repelling or neutral. Moreover, we describe the Siegel discs of the system, since the structure of the orbits of the system is related to the geometry of the *p*-adic Siegel discs. For the case $|a|_p > 1$, we establish that the dynamical system is conjugate to the shift of symbolic dynamics.

## خلاصة البحث


في تطبيقات أرقام *p-adic* قي الفيزياء الرياضي، الطاقة الأبدية تهيج الفوائد المتزايدة في دراسة نظام *p-adic* الديناميكي. من بين تلك الفوائد خريطة *p-adic* اللوجستية. وهي معروفة لدى الجميع بأنها فوضوية. في الدراسة الراهنة نتأمل تعميمها المكعب. ذلك بأن ندرس نظام $f(x) = ax(1 - x^2)$الديناميكي. هذه الرسالة خصصت لتفتيش أثر النظام المذكور. نبحث النظام الديناميكي للوجستية المعمَّمة بالنظر إلى مقياس a . لمعرفة قيمة a ، نتأمل ثلاث قضايا: $|a|_p < 1$ و$|a|_p > 1$ و $|a|_p = 1$. لكل قضية نبحث عن وجود النقطة الثابتة. بجانب ذلك، نبحث أيضا عن نقطتين زمانيتين لقضية $|a|_p < 1$. ليس ذلك كله، بل يتعمق بحثنا في صفات تلك القضايا الثلاث سواء كانت جذابة أو دافعة أو طبيعية. بالإضافة إلى ذلك، نصف أقراص سيجل ( The Siegel discs ) لذلك النظام، نظرا بأن بناء فلك ذلك النظام متصل بهندسة أقراص سيجل ل *p-adic* . لقضية $|a|_p > 1$ نضع أساسا بأن النظام الديناميكي متزاوج لحيلة الديناميكية الرمزية.

# APPROVAL PAGE

I certify that I have supervised and read this study and that in my opinions; it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science (Computational And Theoretical Sciences).

...............................................................
Farrukh Mukhamedov
Supervisor

……...................................................
Pah Chin Hee
Co-Supervisor

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science (Computational And Theoretical Sciences).

..............................................................
Inomzhon Ganiev
Examiner

This thesis was submitted to the Department of Computational and Theoretical Sciences and is accepted as a fulfillment of the requirement for the degree of Master of Science (Computational And Theoretical Sciences).

...............................................................
Samsun Baharin Mohamad
Head, Department of Computational and Theoretical Sciences

This thesis was submitted to the Kulliyyah of Science and is accepted as a fulfilment of the requirement for the degree of Master of Science (Computational And Theoretical Sciences).

.............................................................
Kamaruzzaman Yunus
Dean,
Kulliyyah of Science

# DECLARATION

I hereby declare that this thesis is the result of my own investigations, exceptwhere otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

Wan Nur Fairuz Alwani Binti Wan Rozali

Signature …………………………………… Date ……………………..

INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA

## DECLARATION OF COPYRIGHT AND AFFIRMATION OF FAIR USE OF UNPUBLISHED RESEARCH



**ON GENERALIZED *P*-ADIC LOGISTIC DYNAMICAL SYSTEM**

I hereby affirm that The International Islamic University Malaysia (IIUM) holds all rights in the copyright of this Work and henceforth any reproduction or use in any form or by means whatsoever is prohibited without the written consent of IIUM. No part of this unpublished research may be reproduced, stored in a retrieval system, or transmitted, in any form or by means, electronic, mechanical, photocopying, recording or otherwise without prior written permission of the copyright holder.

Affirmed by Wan Nur Fairuz Alwani Binti Wan Rozali

……………………………. ………………..

Signature Date

# ACKNOWLEDGEMENTS

My utmost gratitude to Allah for His guidance, and giving me strength and courage throughout my life, especially during hard days of my study.

This work would have been possible without the advice and help from my supervisor Assoc. Prof. Dr Farrukh Mukhamedov for giving me the opportunity to work in this interesting field of dynamical system, his invaluable advice, many valuable suggestions, his continual guidance and support, both academically and personally throughout this research.

My sincere gratitude also goes to Dr. Mansoor Saburov for his valuable support, advice, knowledge and suggestions while doing this study.

This work could not have been possible without the help from my father, Assoc. Prof. Dr Wan Rozali b. Wan Hussin and mother, Mdm. Wan Lijah bt Wan Daud, my precious siblings, Wan Ahmad Nor Farihan and Wan Ahmad Nor Fawzul Hakiimie (brothers), Wan Nur Fasihah Zarifah, Wan Nur Fatin Afifah and Wan Nur Fitrihanis Suhaili (sisters). My sincere appreciation to my friends, Bro Ashraf Nawi, Sis Nurul Akma Supar and Sis Siti Fatimah Zakaria in Computational and Theoretical Sciences (CTS) Department for their generous assistance.

Not forgetting my sincere gratitude to MOHE grant FRGS0409-109 for the financial support in the period of completing my thesis and to all the staffs in the CTS Department, Head of Department Dr Samsun Baharin, my Ergodic Theory lecturer, Prof Nasir Ganikhodjaev and Dr Pah Chin Hee in providing me the valuable resources in completing my thesis.

# TABLE OF CONTENTS

# LIST OF TABLES

# LIST OF ABBREVIATIONS

| | |
|---|---|
| $\mathbb{R}$ | Real numbers |
| $\mathbb{N}$ | Natural numbers |
| $\mathbb{Z}$ | Integer numbers |
| $\mathbb{Z}_p$ | Integer numbers on *p* |
| $\mathbb{Z}_p{}^*$ | *P*-adic unit |
| $\mathbb{Q}$ | Rational numbers |
| $\mathbb{Q}_p$ | Rational numbers on *p* |
| $\Sigma$ | Summation |
| $\cap$ | intersection |
| $\subset$ | Subset |
| $Fix\{\}$ | Set of fixed points |
| $Per\ \{\}$ | Set of periodic points |
| $i.e$ | which means |

# LIST OF SYMBOLS

<u>Symbols</u>

| | |
|---|---|
| $a$ | parameter |
| $F$ | field |
| $\|a\|_p$ | norm of parameter $a$ with respect to prime $p$ |
| $(X, d)$ | Metric space |
| $+,-,\cdot$ | Plus, minus, multiplication |
| $d$ | metric |
| $\|\|x\|\|$ | Norm of $x$ |
| $ord_p a$ | Highest order of integer $a$ with respect to prime $p$ |
| $\equiv$ | Congruent |
| $x^{(0)}$ | fixed point |
| $\cup$ | union |
| $\alpha$ | alpha |
| $\beta$ | Beta |
| $\tau$ | tau |
| $\circ$ | composition |
| $x_1$ | value of $x_1$ |
| $x_{2/3}$ | value of $x_2$or $x_3$ |
| SI | Siegel disc |

# CHAPTER ONE

# INTRODUCTION

## 1.1 LITERATURE REVIEW

Over the last century, *p*-adic numbers and *p*-adic analysis have come to play major role in the number theory. *P*-adic numbers were first introduced by a German Mathematician, K. Hensel. During a century after their discovery they were considered mainly objects of pure mathematics. Starting from 1980's various models described in the language of $p$-adic analysis have been actively studied. *P*-adic numbers have been widely used in many applications mostly in $p$-adic mathematical physics, quantum mechanics and others which rose the interest in the study of $p$-adic dynamical systems.

One of the famous investigation in the area of *p*-adic analysis is devoted to the investigation of quantum mechanics models using mathematical physics equations which can referred in Aref'eva, Dragovic and Volovich (1988) and Mahler (1973). In addition one of the applications of *p*-adic numbers in quantum physics most appear in quantum logic in Betrametti and Cassinelli (1972). There are also numerous applications of the *p*- adic analysis to mathematical physics have been proposed and can referred to Avetisov, Bikulov and Kozyrev (1999) and Khrennikov (1996).

The main reason of studying dynamical systems is to predict the future of a given phenomenon. Even we could not exactly guarantee its liability, however we are often able to get some information about long time behavior of the system. The most used set of numbers is the field of rational numbers, $\mathbb{Q}$, and we often use the absolute value of the difference of two measures for distance. But, there are many other possibilities, for example, the so called $p$-adic absolute value.

On the other hand, the study of $p$-adic dynamical systems arises in Diophantine geometry in the construction of canonical heights, used for counting rational points on algebraic vertices over a number field can be found in Call and Silverman (1993). In Khrennikov, Yamada and van Rooji (1999) and also Thiran, Verstegen and Weters (1989), the $p$-adic field has arises in physics in the theory of superstrings, promoting questions about dynamics. Also some applications of $p$-adic dynamical systems to some biological, physical systems were proposed in the study of Albeverio, Kloeden and Khrennikov (1998,1999) and Avetisov, Bikulov and Kozyrev (1999). Not only that, in Batra and Morton (1994) and Lindahl (2004), the dynamical systems (not only monomial) over finite field extensions of the $p$-adic numbers were considered. Other studies of non-Archimedean dynamics in the neighborhood of a periodic and of the counting of periodic points over global fields using local fields appeared in Gundlach, Khrennikov and Lindahl (2001) and Hsia (2000). Certain rational $p$-adic dynamical systems were investigated in Khamraev and Mukhamedov (2006) and Mukhamedov (2007) which appear from problems of $p$-adic Gibbs measures (see Khamraev, Mukhamedov and Rozikov (2004), Mukhamedov and Rozikov (2004), Mukhamedov and Rozikov (2005) and also Mukhamedov, Rozikov and Mendes (2006)). Note that in Benedetto (2003) and Bézivin (2004), a general theory of $p$-adic rational dynamical systems over complex $p$-adic field $\mathbb{C}_p$ has been developed. In Benedetto (2000 and 2001),the Fatou set of a rational function defined over some finite extension $\mathbb{Q}_p$ has been studied. Besides, an analogue of Sullivan's no wandering domains theorem for $p$-adic rational functions, which have no wild recurrent Julia critical points, was proved.

The most studied discrete $p$-adic dynamical systems (iterations of maps) are the so called monomial systems. In Albeverio, Khrennikov, Tirozzi and Smedt (1998) and Khrennikov (1996), the behavior of a $p$-adic dynamical system $f(x) = x^n$ in the fields of $p$-adic numbers $\mathbb{Q}_p$ and $\mathbb{C}_p$ was investigated. In Khrennikov and Nilsson (2004) perturbated monomial dynamical systems defined by functions $f_q(x) = x^n + q(x)$, where the perturbation $q(x)$ is a polynomial whose coefficients have small $p$-adic absolute value, have been studied. It was investigated the connection between monomial and perturbated monomial systems. Formulas for the number of cycles of a specific length to a given system and the total number of cycles of such dynamical systems were provided. Even for a quadratic function $f(x) = x^2 + c, c \in \mathbb{Q}_p$, its chaotic behavior is complicated (see Albeverio, Kloeden and Khrennikov (1998) , Shabat (2004), Thiran, Verstegen and Weters (1989) and Woodcock and Smart (1998)). In Dremov, Shabat and Vymova (2006) and Shabat (2004) the Fatou and Julia sets of such a $p$-adic dynamical systems were found. Certain ergodic and mixing properties of monomial and perturbated dynamical systems have been considered in Anashin (2006) and Gundlach, Khrennikov and Lindahl (2001).

In Khrennikov (1997), they proposed to apply $p$-adic dynamical systems for modeling of cognitive processes. In applications of $p$-adic numbers to cognitive science, the crucial role is played not by algebraic structure of $\mathbb{Q}_p$, but by its tree-like hierarchical structure where the structure of a $p$-adic tree is used for hierarchical coding of mental information and the parameter p characterizes the coding system of a cognitive system.

Ergodicity of monomial $p$-adic dynamical systems was studied in Gundlach et al (2001). As expected in Khrennikov (1997) also, nontrivial dependence of non/ergodicity on the parameter *p*, was found in Khrennikov and Nilsson (2004) Investigation on *p*-adic ergodicity induced interest in the problem of existence of conjugate maps for *p*-adic (analytic) dynamical systems and the problem of small denominators in the *p*-adic case (see Fan, Li, Yao and Zhao (2007)). Again in Khrennikov and Nilsson (2004), they performed detailed investigations to find the best possible estimates for "small denominators" and radii of convergence for analytic conjugate maps.

There is also one recent study using *p*-adic dynamical systems in cryptography. Recently, T-functions were found to be useful tools to design fast cryptographic primitives and ciphers based on usage of both arithmetic (addition, multiplication) and logical operations, (see Hong, Lee, Yeom and Han (2005), Kai-Thorsten (2007) and Woodcock and Smart (1998). In Anashin, Khrennikov and Yurova (2011), it provides a new criteria for bijectivity/transitivity of T-functions. In their opinion, these new criteria might be better applicable to T-functions that are represented via compositions of standard computer instructions on representation of T-functions via additions of some non-negative constants. The representation is based on van der Put series which is special convergent series from *p*-adic analysis which explains such *p*-adic analysis and *p*-adic ergodic theory are the mathematical tools in their investigation. In that paper also mentioned that *p*-adic ergodic theory is a crucial part of non-Archimedean dynamics and it has been rapidly developing in many disciplines of science such as computer science, molecular biology, cognitive sciences and others (see also Woodcock and Smart (1998)).

Above all, one of the simplest and interesting perturbated dynamical systems is the so called the logistic maps $f(x) = cx(1-x)$ and generalized logistic maps which are well known in the literature and it is one of a great importance in the study of dynamical systems where it can be referred to Aulbach and Kininger (2004), Devaney (1989) and Jaganathan and Sinha (2005). A $p$-adic analogues of the logistic map where considered in Dremov, Shabat and Vymova (2006), Fan, Liao, Wang and Zhao (2007) and Shabat (2004) it is shown such a map is chaotic. In Mukhamedov and Mendes (2007), it is considered a generalized logistic map of the form $G_b(x) = (ax)^2\,(x+1)$. It was investigated the asymptotic behavior, attractors, Siegel discs of the mentioned dynamical system. In the present thesis, we are going to consider another kind of generalization of the logistic dynamical system namely we consider dynamical systems given by $f_a(x) = ax(1-x^2)$ which is not conjugate to the $G_b(x)$ (see chapter 2 for more details).

## 1.2 OBJECTIVES OF THE THESIS

In this thesis, we will investigate another generalized cubic logistic function $f_a(x) = ax(1-x^2)$. Our objectives of the study are as follows:

1. Description of all fixed points of the dynamical system with respect to parameter $a$.
2. Description of the behavior of the fixed points of the given dynamical system with respect to parameter $a$.
3. Investigation of attractors and Siegel discs of dynamical system.
4. Investigation of dynamics where the fixed points are repelling.
5. Description of all periodic points for certain values of parameter $a$.

Let us briefly emphasize how to carry our investigation according to the objectives. To describe fixed points, we first formally find the fixed points (respectively periodic points) and then we provide conditions for the parameter $a$ to ensure the existence of such points since not every polynomial equation over $\mathbb{Q}_p$ may have solution.

After this, we turn to the investigation of behavior of these fixed points and this study allows us to determine attractors and Siegel discs of such points. Note that globally attracting sets play an important role in dynamics, restricting the behavior to certain regions of the phase space. However, the global attracting sets described can be complicated as it may contain chaotic dynamics. Therefore, in the thesis we will describe the basin of attraction of such dynamical system. Moreover, we also describe the Siegel discs of the system, since the structure of the system orbit is related to geometry of the $p$-adic Siegel discs.

To investigate the dynamics of $f(x)$ where the fixed points are repelling, we follow the paper by Fan, Liao, Wang and Zhao (2007) and we show for certain values of parameter $a$, the given dynamical system is conjugate to the shift of the symbolical dynamical system. Moreover, we also study the behavior of periodic points.

## 1.3 PLANNING OF THE THESIS

In Chapter 1 after the introduction, we provide necessary information about $p$-adic numbers and the dynamical system which will be used in our further investigation.

Chapter 2 consists of two sections. The first section is devoted to the conjugacy of a function in previous study with this new function which will be investigated. The next section is the investigation of the existence of the fixed points and we consider all possible cases of $|a|_p$.

In Chapter 3, we describe the behavior of the fixed points whether they are attracting, repelling or neutral. Next, in Chapter 4, the size of attractors and Siegel discs of fixed points are described. We investigate in Chapter 5 the existence and the dynamics of 2-periodic points. Moreover, the behavior and the size attractors of such periodic points were investigated.

In the final Chapter 6, we concentrate ourselves to the case $|a|_p > 1$. In this case, all fixed points are repelling and therefore, we follow paper Fan, Liao, Wang and Zhao (2007) and establish conjugacy of the dynamical system to the shift of the symbolic dynamical system.

## 1.4 *P*-ADIC NUMBERS

If $X$ is a nonempty set, a distance, or metric, on $X$ is a function $d$ from pairs of elements $(x, y)$ of $X$ to the nonnegative real numbers such that

*(i)* $d(x, y) = 0$ *if and only if* $x = y$

*(ii)* $d(x, y) = d(y, x)$

*(iii)* $d(x, y) \leq d(x, z) + d(x, z)$ *for all* $z \in X$

A set $X$ together with a metric $d$ is called a metric space. The same set $X$ can give rise to many different metric spaces$(X, d)$. Recall that a field $F$ is a set together with two operations $+$ and $\cdot$ such that $F$ is commutative group under $+, F - \{0\}$ is commutative group under $\cdot$, and the distributive law holds. The examples of a field to have in mind at this point are the field Q of rational numbers and the field R of real numbers.

The metrics $d$ will come from norms on the field $F$ means that a map denoted $\|\cdot\|$ from $F$ to the nonnegative real numbers such that

*(i)* $\|x\| = 0$ *if and only if* $x = 0$

*(ii)* $\|x \cdot y\| = \|x\| \cdot \|y\|$

*(iii)* $\|x+y\| \leq \|x\| + \|y\|$

Metric $d$ "comes from" (or "induced by") a norm $\|\cdot\|$, means that $d$ is defined by : $d(x,y)=\|x-y\|$ and a basic example of norm on the rational number field $\mathbb{Q}$ is the absolute value $|x|$ where the induced metric $d(x,y)=|x-y|$ is the usual concept of distance on the number line.

Here are some useful definitions and notions that might be useful in this whole thesis.

**Definition 1.1.** *For any nonzero integer $a$, let $ord_p a$ be the highest power of $p$ which divides $a$, i.e., the greatest $m$ such that $a \equiv 0 (\bmod p^m)$. (The notation $a \equiv b (\bmod c)$ means: c divides $a-b$.)*

**Examples**.

$$ord_5 35 = 1, ord_5 250 = 3, ord_2 96 = 5, ord_2 97 = 0 \tag{1.1}$$

(If $a=0$, we agree to write $ord_p 0 = \infty$). Note that $ord_p$ behaves a little like logarithm would : $ord_p a_1 a_2 = ord_p a_1 + ord_p a_2$.

Therefore, we define a map $|.|_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = \begin{cases} \dfrac{1}{p^{ord_p x}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \tag{1.2}$$

**Proposition 1.2.** $|\cdot|_p$ *is a norm on* $\mathbb{Q}$.

The full proof can be found in Koblitz (1977) and Katok (2001).

**Definition 1.3.** *A norm is called non-Archimedean if* $|x+y| \le \max(|x|,|y|)$ *always holds. A metric is called Archimedean if* $d(x,y) \le \max(d(x,z), d(z,y))$; *in particular, a metric is non-Archimedean if it is induced by a non-Archimedean norm, since in that case* $d(x,y) = |x-y| = |(x-z)+(z-y)| \le \max(|x-z|,|z-y|) = \max(d(x,z), d(z,y))$. *Thus,* $|\cdot|_p$ *is a non-Archimedean norm on* $\mathbb{Q}$.

Some properties of the non-Archimedean metric $|\cdot|_p$ seems very strange. Here is one example :

For any metric, Property (3): $d(x,y) \le d(x,z) + d(z,y)$ is known as the "triangle inequality," because in the case of the field C of complex numbers (with metric $d(a+bi, c+di) = \sqrt{(a-c)^2 + (b-d)^2}$ ), it says that in the complex plane the sum of two sides of a triangle is greater than the third.

However, in the non-Archimedean norm on a field $F$ is not the same as usual Archimedean norm. For simplicity, we suppose $z = 0$. Then the non-Archimedean triangle inequality says that $|x-y| \le \max(|x|,|y|)$. Suppose $x$ and $y$ have different length, for instance $|x| \le |y|$. Then, the third side $x-y$ has length

$$|x-y| \le |y|. \tag{1.3}$$

But

$$|y| = |x-(x-y)| \le \max(|x|,|y|). \tag{1.4}$$

Since $|y| \ge |x|$ then we must have $|y| \le |x-y|$, and so $|y| = |x-y|$. Thus, if the two sides $x$ and $y$ are not equal in length, the longer of the two must have the same length as the third side or in other words, every triangle is "isosceles".

In the case of $|\cdot|_p$ on $\mathbb{Q}$, it says that if two rational numbers are divisible by different powers of $p$, then their difference is divisible precisely by the lower power of $p$ (which is what it means to be the same "size" as the bigger of the two.)

**Theorem 1.4.** *(Ostrowski). Every nontrivial norm* $|\cdot|$ *on* $\mathrm{Q}$ *is equivalent to* $|\cdot|_p$ *for some prime* $p$ *or for* $p=\infty$. *It has the form of*

$$|x|_p = \frac{1}{p^{ord_p(x)}},\ x \in \mathrm{Q} \quad and \quad |x|_\infty = \begin{cases} x, & x \neq 0 \\ -x, & x = 0 \end{cases}.$$

Note that "trivial" norm means the norm $|\cdot|$ such that $|0|=0$ and $|x|=1$ for $x \neq 0$.

**Theorem 1.5**. $\bar{F}$ *is a field.*

**Definition 1.6.** *For any* $A \in \bar{F}$ *put*

$$\|A\| = \lim_{n\to\infty} \|a_n\|,$$

*where* $\{a_n\}$ *is any Cauchy sequence in A.*

We define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $p$-adic norm $|.|_p$in (1.2). The $p$-adic norm is extended to $\mathbb{Q}_p$ according to Definition 1.6 and $(\mathbb{Q}_p, |.|_p)$ is a complete normed field. We call $\mathbb{Q}_p$ the field of $p$-adic numbers. The elements of $\mathbb{Q}_p$ are equivalent clasess of Cauchy sequence in $\mathbb{Q}$ with respect to the extension of the $p$-adic norm.

For some $a \in \mathbb{Q}_p$ let $\{a_n\}$ be a Cauchy sequence of rational numbers representing $a$. Then by definition

$$|a|_p = \lim_{n\to\infty} |a_n|_p$$

Therefore, the set of values that $|.|_p$ takes on $\mathbb{Q}_p$ is the same as it takes on $\mathbb{Q}$, namely $\{p^n, n \in \mathbb{Z}\} \cup \{0\}$.

Each equivalence class of Cauchy sequences defining some element of $\mathbb{Q}_p$ contains a unique canonical representative Cauchy sequence. In order to describe its construction, we need the following Lemma.

**Lemma 1.7.** *If $x \in Q$ and $|x|_p \leq 1$, then for any $i$, there exists an integer $\alpha \in Z$ such that $|\alpha - x|_p \leq p^{-i}$. The integer $\alpha$ can be chosen in the set $\{0,1,2,\cdots,p^i - 1\}$, and is unique if chosen in this range.*

**Theorem 1.8.** *Every equivalance class $a$ in $Q_p$ satisfying $|a|_p \leq 1$ has exactly one representative Cauchy sequence $\{a_i\}$ such that:*

*(i)* $a_i \in Z$, $0 \leq a_i < p^i$ *for* $i = 1, 2, \cdots$

*(ii)* $a_i \equiv a_{i+1} (\text{mod } p^i)$ *for* $i = 1, 2, \cdots$

We can find the full proof of this Theorem 1.8 in the book of Katok (2001) and Robert (2000).

If $a \in Q_p$ with $|a|_p \leq 1$, then it is convenient to write all the terms $a_i$ of the representative sequence given by the previous theorem in the following way

$$a_i = d_0 + d_1 p + \cdots + d_{i-1} p^{i-1}, \tag{1.5}$$

where all the $d_i s$ are integers in $\{0,1,2,\cdots,p-1\}$. Our condition (ii) precisely means that

$$a_{i+1} = d_0 + d_1 p + \cdots + d_{i-1} p^{i-1} + d_i p^i, \tag{1.6}$$

where the "$p$-adic digits" $d_0$ through $d_{i-1}$ are all the same as for $a_i$. Thus $a$ is represented by the convergent (in the $p$-adic norm, of course) series

$$a = \sum_{n=0}^{\infty} d_n p^n, \tag{1.7}$$

We will write

$$a = \cdots d_n \cdots d_2 d_1 d_0 \tag{1.8}$$

which can be thought of as a number, written in the base $p$, that extends infinitely far to the left, or has infinitely many $p$-adic integers and call this the canonical p-adic expansion or canonical form of *a*.

If $|a|_p > 1$, then we can multiply $a$ by a power of $p$ (namely by $p^m = |a|_p$) so as to get $p$-adic number $a' = ap^m$ that does satisfy $|a'|_p \leq 1$. Then we can write

$$a = \sum_{n=-m}^{\infty} d_n p^n, \tag{1.9}$$

where $d_{-m} \neq 0$ and $d_i \in \{0,1,2,\cdots,p-1\}$, and represent the given $p$-adic number $a$ as a fraction in the base $p$ with infinitely many $p$-adic digits before the point and finitely many digits after:

$$a = \cdots d_n \cdots d_2 d_1 d_0 \cdot d_{-1} \cdots d_{-m}; \tag{1.10}$$

this representation is called the canonical $p$-adic expansion of $a$. Note that for negative canonical form, it can be written in this way:

$$\begin{aligned} -a &= -d_0 - d_1 p - d_2 p^2 - \cdots \\ &= (p - d_0) + (p - d_1 - 1)p + (p - d_2 - 1)p^2 + \cdots \end{aligned} \tag{1.11}$$