RANDOM BINOMIAL TREE MODELS AND PRICING OPTIONS

BY

KAMOLA BAYRAM

A thesis submitted in fulfilment of the requirement for the degree of Master of Science

Kulliyyah of Science
International Islamic University Malaysia

JULY 2013

ABSTRACT

The binomial tree model is a natural bridge, overture to continuous models for which it is possible to derive the Black-Scholes option pricing formula. In turn a binomial branch model is the simplest possible non–trivial model which theory is based on the principle of no arbitrage works. The binomial tree model is defined by a pair of real numbers (*u,d*) such that the stock can move up from S_0 to a new level, uS_0 or down from S_0 to a new level, dS_0, where $u > 1$; $0 < d < 1$. We shall call pair (*u,d*) the environment of the binomial tree model. The binomial tree model is called *a random binomial tree model*, if the corresponding environment is random. We introduce a simplest random binomial tree model, illustrating that risk – neutral valuation gives the same results as no-arbitrage arguments and describe some properties of the random binomial tree models. The random binomial tree model produces results which are a reflect of the real market better than the binomial tree model when fewer time steps are modelled. The model is solvable and there exist analytic pricing formulae for various options. In this thesis we produce these formulas for a European call and put options and also an American call and put options for a single period, a two periods and an arbitrary *N*-period time steps.

ملخص البحث

نموذج الشجرة ذو الحدين هو الجسر الطبيعي, والنافذة للنماذج المتصلة التي يمكن اشتقاق صيغة التسعير بلاك- سكولز, وبدوره نموذج الفرع ذو الحدين هو ابسط نموذج غير بديهي محتمل والذي يعتمد على أساس عدم مرجحة الأعمال. نموذج الشجرة ذو الحدين معرف باستخدام زوج من الأعداد الحقيقية (u,d) بحيث أن المخزون يستطيع أن يرتفع من S_0 إلى مستوى جديد uS_0 ا و ان ينزل من S_0 إلى مستوى جديد dS_0, حيث أن $u>1;\ 0<d<1$ سوف نطلق على الزوج (u,d) البيئة لنموذج الشجرة ذو الحدين, نموذج الشجرة ذو الحدين يسمى عشوائي إذا كانت البيئة المتناظرة عشوائية, سوف نقدم ابسط نموذج شجرة ذو حدين عشوائي, توضيح المخاطر, التقييم المحايد يعطي نفس النتائج مثل متطابقات عدم المراجحة ويصف بعض خصائص نموذج الشجرة ذو الحدين العشوائي, نموذج الشجرة ذو الحدين العشوائي يعطي نتائج أكثر دقه بالمقارنة مع نموذج الشجرة ذو الحدين عندما تتنمذج خطوات وقت قليلة. النموذج قابل للحل ويوجد صيغة تسعير تحليلية لعدة خيارات. في هذه الأطروحة نقدم هذه الصيغة لخيارات الاتصال الاوروبي وايضا والاتصال الامريكي ونضع خيارات للتناوب المنفرد والتناوب الثنائي والتناوب **N** لخطوات الوقت.

.

APPROVAL PAGE

I certify that I have supervised this study and that in my opinion, it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

……….......…………………
Mohd Islam Aminul
Supervisor

…………………......…………
Pah Chin Hee
Co-Supervisor

I certify that I have read this study and that in my opinion, it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

....…..……..………..….......…

Examiner

This thesis was submitted to the Department of Computational and Theoretical Sciences and is accepted as a fulfilment of the requirement for the degree of Master of Science.

....…..……......………………
Pah Chin Hee
Head, Department of
Computational and
Theoretical Sciences

This thesis was submitted to the Kulliyyah of Science and is accepted as a fulfilment of the requirement for the degree of Master of Science.

…………………………........
Kamaruzzaman Yunus
Dean, Kulliyyah of Science

DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions

Kamola Bayram

Signature……………...……….   Date………....……….......

INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA

DECLARATION OF COPYRIGHT AND AFFIRMATION OF FAIR USE OF UNPUBLISHED RESEARCH

RANDOM BINOMIAL TREE MODELS AND PRICING OPTIONS

I hereby affirm that The International Islamic University Malaysia (IIUM) holds all rights in the copyright of this Work and henceforth any reproduction or use in any form or by means whatsoever is prohibited without the written consent of IIUM. No part of this unpublished research may be reproduced, stored in a retrieval system, or transmitted, in any form or by means, electronic, mechanical, photocopying, recording or otherwise without prior written permission of the copyright holder.

Affirmed by Kamola Bayram

……………………………. ………………..
Signature Date

Dedicated to

my parents

my family

ACKNOWLEDGEMENTS

My utmost gratitude to Allah for His guidance, and giving me strength and courage throughout my life, especially during hard days of my study.

My sincere appreciation to my supervisors Asst. Prof. Dr Aminul Mohd Islam and Asst. Prof. Dr. Pah Chin Hee for giving me the opportunity to work in this interesting field of financial mathematics, their invaluable advice, many valuable suggestions, continual guidance and support, both academically and personally throughout this research.

This work could not have been possible without the help from my father and mother. My sincere appreciation to my friends, Bro Izzat Qaralleh, Sis Wan Nur Fairuz Alwani Bt Wan Rozali and Sis Siti Fatimah Zakaria in Computational and Theoretical Sciences (CTS) Department for their generous assistance.

Not forgetting my sincere gratitude to IIUM Grant EDW B 12-403-0881 and for IIUM Postgraduate Niche Area Scholarship IIUM/503/SA/12/10/2/4 for the financial support during the period of completing my thesis and to all the staffs in the CTS Department, Heads of Department Dr. Samsun Baharin and Dr. Pah Chin Hee in providing me the valuable resources in completing my thesis.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF COMMON SYMBOLS / ABBREVIATIONS

S_0 – The initial price of a stock

S_T – The price of a stock at time *T*

K – Strike price (Exercise price)

c – Value of a European call

C – Value of an American call

p – Value of a European put

P – Value of an American put

r – Risk free interest rate

T – Time to maturity

CHAPTER ONE

INTRODUCTION

1.1 LITERATURE REVIEW

Human being always wants to avoid the unknown. This rule is valid today as well as it was valid hundreds of years ago. One of the best ways of avoiding the unknown is hedging against risk. Hedging was very important for farmers, since their harvest was under risks such as drought, flood and other kinds of natural disasters.

The first primitive options were used in Greece to hedge against the low prices of olives during harvest (Schoutens 2003). In the early 1600s the tulip trading became popular in Netherlands and other countries, for example 1600s were known as *"Lale Devri"* which means "Time of Tulips" in Ottoman Empire. These developments require using of option contracts since both wholesalers and tulip growers wanted to protect themselves against an unknown risk that is likely to happen during the harvest. Even though the trading of options on tulip bulbs became very popular they were far from standardization and regulation. At the second half of 17th century the option market was collapsed. Bad reputation of option contacts last for another three centuries.

Chicago Board Options Exchange in 1973 started a call option trading on 16 underlying stocks. In 1977 the range of underlying stocks was widened to 43 and trading in put options begun. As reported in the official website of the Bank for International Settlements, OTC derivatives amounted to $639 trillion at the end of June 2012. (http://www.bis.org)

In the efficient market, prices of securities respond immediately and properly to the related information. The randomness of information implies random fluctuation of stock prices. Therefore, the share prices cannot follow any particular pattern but fluctuate in an unpredictable random manner.

One way of establishing the randomness of changes in stock prices is to look at them through the methods of statistics and demonstrate that they behave in the same way as any other random variables. Bachelier (Bachelier 1900(a), 1900 (b); an English translation is available in Gootner 1964) formulated Brownian motion for mathematics, five years before Einstein's famous paper (Einstein 1905). With a mathematical rigor, given his assumptions, Bachelier established the idea that financial assets prices change randomly and could be described as following a normal distribution. This means that movements of those asset prices could not be predicted by precision. Nevertheless, they move in a way that could be analysed using statistics. Bachelier's article (Bachelier 1900(a)) was based on his doctoral dissertation written at the University of Paris in 1900 under the direction of the great mathematician, Henri Poincare (Stabile 2005).

Bachelier had a key insight into what made security prices unpredictable. Every sale of a security is also a purchase. This implies that investors who buy a security expecting an increase of prices whereas investors who sell a security have an opposite expectation. Given this divided opinion among investors, any slight change in information of investor outlook can change the price in either direction. An increase is just as likely as a decrease, and no one could predict which would take place (Bernstein 1992).

Bachelier's work was not much appreciated in that day. Poincare did not grade his dissertation as earning the highest honours. He felt that its topic was too removed from that statisticians and probability theorists should be studying. Peter Bernstein, in his history of investment methods, tells us that Bachelier's dissertation and a small book he published on it in 1914 vanished from the academic discussion until L.J. Savage, a statistician at the University of Chicago, accidentally found the book at a library in the 1950's.

Savage was impressed and spread the word about Bachelier to economist friends. One of them, Paul Samuelson, was more impressed, and made Bachelier well known throughout the economics profession (Stabile 2005).

The Bachelier's version of the random walk hypothesis was crude by today's standards. Osborne (1959), Moore (1960), Alexander (1961, 1964), Granger and Morgenstern (1963), Kendall (1953), and Samuelson (1965) modified the Bachelier's model (also known as "arithmetic Brownian motion" model) to geometric random walk.

A binomial tree model for option pricing was pioneered by Cox, Ross and Rubinstein (1979) and is one of the most successful models dealing with derivative asset valuation. The model was first proposed for the valuation of stock options. It was then extended to the valuation of several derivative assets with complex pay-offs and different underlying assets. Although not as instantaneously recognized as the Black-Scholes (1973) option pricing model, binomial tree model is more easily generalizable and it is often able to handle a variety of conditions for which the Black-Scholes model cannot be applied.

Cox, Ross and Rubinstein's proved that their model merges into the Black-Scholes model when the time steps between successive trading instances approach zero. The main property of their model is that it is consistent with the standard Black-Scholes formula for European options. A discrete time framework used in the binomial tree model to detect the evolution of the key variable upon which the claim of interest is positive contingent. To value the option at each node of the tree Cox, Ross and Rubinstein (1979) developed a risk neutral probability.

Due to versatility of the tree model a number of extensions to the basic model have been published. Randleman and Barter (1979) applied the binomial tree model for the pricing of options on debt instruments. Boyle (1988) developed a trinomial tree, which is generalization of a binomial model. Trigeorgis (1993) considered the binomial tree model to value investments with various options. Tian (1993) considered special classes of binomial and trinomial models where the models parameters came as unique solutions to equation system. Hull and White (1988) proposed a totally different and interesting approach to improve the accuracy of binomial models. They transferred the control-variate technique from the Monte Carlo method. Boyle's (1988) approach was modified by Yisong (1993) by presenting a general methodology that can be applied to any multi-dimensional tree approach. He proposed a modified approach to the selection of tree parameters including probabilities and jumps using additional restrictions. Sandman et al. (1993) considered a model to price European options assuming that the interest rate is random. Rubinstein (1994) developed a new method for inferring risk-neutral probabilities of option prices from observed market prices. These probabilities used to infer a binomial tree by implementing a simple backward recursive procedure. Jabbour et al. (2005) suggested a new tree model for developing an n-order multinomial tree parameterization for a single-state option pricing model. Merton (1973), Black (1976) and Barone-Adesi and Whaley (1987), among others, showed that futures contracts, stock index options and currency options may be assimilated to options on a stock that pays a continuous dividend. Arnold, Crack and Schwartz

(2010) who generalized the Rubinstein (1994) risk neutral implied binomial tree model by introducing a risk premium. One of the latest extensions was done by Bayram (2013) who studied a trinomial tree by randomizing the jump sizes of the tree.

To calculate various types of options on a tree there were introduced numbers of new approaches. Cox and Rubistein (1985) developed a model to value down-and-out call options for some particular cases. Hull and White (1993) improved the initial binomial tree model to price some exotic options. Rubinstein (1998), Yamada and Primbs (2001), Amin (1993) and Boyle (1988) constructed trees for a multi underlying assets. Cheuk and Vorst (1994) studied a model where it is considered the path dependency of look-back options. Rubinstein (1994), Derman and Kani (1994a) and Dupire (1994) used tree models to investigate implied volatility surfaces. The method was later discussed and extended by Barle and Cakici (1995), Derman and Kani (1994b), Chriss (1996), Buchen and Kelly (1996), Jackwerth and Rubinstein (1996), and others.

The implied tree models are a discretization of a stock price process where the local volatility is a function of both the price level of the underlying asset and time. The model can be used to price exotic options and other derivatives. The attraction of the binomial tree lies not in its ability to replicate the Black – Sholes model, but in its ability to handle more complex options.

1.2 OBJECTIVES OF THE THESIS

In this thesis we introduce and study a new tree model which we call "Random binomial tree model". For the usual binomial tree model, during each time step the value of stock either moves up with a certain probability by u times or moves down by d times with a certain probability. We will call the pair (u, d) the environment of a binomial tree model. Note that in this model the pair (u, d) is the same for any moment of time.

Now we define a random environment and a random binomial tree model as follows. Let $\{U_n\}$ and $\{D_n\}$ be the sequences of independent, identically distributed random variables with $U_n > 1$ and $0 < D_n < U_n$. This pair (U_n, D_n) is called a random environment. The random binomial tree model in a random environment (U_n, D_n), is defined such that during the n-th moment of time, the value of the stock either moves up with a certain probability of U_n times or moves down by D_n times with a certain probability for some realization of these random variables. If the random variables U_n and D_n are constant, i.e., taking a single value $U_n = u$ and $D_n = d$, then we have the usual tree model.

In this thesis, we consider the simplest random binomial tree model when the random variables $\{U_n\}$ and $\{D_n\}$ take two values, u_1, u_2 and d_1, d_2 respectively. Thus, the pair of random variables (U_n, D_n) describes the two possible environments (u_1, d_1) and (u_2, d_2). To avoid complicated mathematical formulations we introduce and explore a simplest example of a random environment in binomial tree model using tossing of a coin.

Let us consider two environments (u_1,d_1) and (u_2, d_2). We will toss repeatedly a coin and if the result is "*Head*" we will apply the first environment and if the result is "*Tail*" we will apply the second one. That is we will choose environment randomly and we will call such model the *simplest random binomial tree model*. The random binomial tree model produces more realistic results than the binomial tree model when fewer time steps are modelled. For these models one can derive pricing formulae for various options. In this thesis we produce some formulae for a European put and call options and also for an American call and put options for a single period, two periods and an arbitrary N-period time steps.

1.3 OVERVIEW

In this work, we introduce binomial tree models which are called random binomial tree models, defined by a random environment. This thesis contains four chapters. The main results of the thesis are given in chapter three.

This thesis is organized as follows: In Chapter one, we provide a literature review on binomial tree models and options and the outline of the objectives of the thesis. Chapter two consists of three sections and devoted to binomial tree models, options and random walk in a random environment. In these sections, we recall some notations, basic definitions and results of binomial tree models, options and random walk in a random environment respectively. In Chapter three, we introduce and study random binomial tree models. This chapter consists of five sections. In the first section, we give the general definition of random binomial tree models and in the second section we select a class of simplest random binomial tree models. In the remaining four sections we study the different options for a single period, two period and an arbitrary N-period models, respectively. Chapter four includes the conclusion.

CHAPTER TWO

PRELIMINARIES

2.1 BINOMIAL TREE MODELS

2.1.1. Mathematical models of financial markets

We start from the simplest mathematical models of financial markets. The results below universal commodity could be found in Black and Scholes (1973), and Pascucci (2011). We consider only two time states: present and future. Let the market has n quantity of stocks with prices $S_i(t)$, $i = 1...n$, $t=0$, 1, where values $S_i(0)$ are known determined constants. Uncertainty of future is defined by final states $j = 1...m$, where one of them certainly will happen, i.e. $S_i(1)$ is a discrete random variable for $i = 1...n$. The future price of stocks can be determined with matrix of payoff $n \times m$.

$$S = \begin{pmatrix} S_{11} & S_{12} & \dots & S_{1m} \\ S_{21} & S_{22} & \dots & S_{2m} \\ \dots & \dots & \dots & \dots \\ S_{n1} & S_{n2} & \dots & S_{nm} \end{pmatrix}$$

where, S_{ij} is the price of a stock i, if in the future j state will come. We set up a portfolio of securities which compose of stocks $\overline{\theta} = (\theta_1, \theta_2, ..., \theta_n)$, where θ_i is the quantity of *i stocks* in portfolio. Need to note that components of this vector are not necessarily positive; negativity of θ_i means that the holder of the portfolio must give back θ_i units of i – stocks.

Let $\overline{g} = (S_1, S_2, ..., S_n)$ be the price vector for given stocks at current period. Then the market price for the current period will be equal to

$\overline{g} * \overline{\theta} = S_1 * \theta_1 + ... + S_n * \theta_n$; and in future will be introduced as a payoff vector by $\overline{\theta} * \mathbf{S}$ portfolio. Since on the market, except risky stocks, yet exists bonds, we will consider that one of the entered stocks, first for instance, is belong to bank account (bond). Let us assume that the price of a bond to be \$1, and in the future \$(1+*r*)*,* where *r > 0* is a bank interest rate.

Any derivatives in this model can be characterized with its own *payoff function* in the future, which depends on a stock price. Since variable states in a single period model *m,* then possible payoff values can be introduced as a payoff vector $\overline{F} = (F_1, F_2, ..., F_m)$, where F_j is a payoff by securities on the *j* state.

Let $\overline{F}$ be the payoff vector of some security. It is known that, between all possible vectors *F* and all possible securities in the market one to one correspondence exists.

Definition 2.1.1. The bond with the $\overline{F}$ payoff vector is called accessible, if one can find portfolio $\overline{\theta}$ with $\overline{F} = \overline{\theta} * \mathbf{S}$.

If the market is complete, i.e. any bond is accessible, and no arbitrage occurs, then the problem of finding the correct price of any bond or stock is solved rather easily. Precisely, let $\overline{F}$ be the payoff vector of bonds, then by virtue of full markets, exists such portfolio $\overline{\theta}$, that $\overline{F} = \overline{\theta} * \mathbf{S}$. To find the necessary portfolio, we just need to solve the linear equations as follows:

$$\sum_{i=1}^{n} S_{ij}\theta_i = F_j, \quad j = 1..m$$

Since there is no arbitrage opportunity exists, the initial value of bonds should be equal to the price of this portfolio, i.e. $\overline{\theta} * \overline{g}$.

There is a more simple way to determine the initial price of the bond without finding portfolio $\bar{\theta}$.

Definition 2.1.2 *Vector* $\overline{\psi} = (\psi_1, \psi_2, ..., \psi_m),\ \psi_j > 0,\ j = 1...m,$ *is called state price vector, if* $\mathbf{S} * \overline{\psi} = \overline{g}$.

If the price vector $\overline{\psi}$ exists, then according to equalities $\sum_{i=1}^{n} S_{ij}\theta_i = F_j,\ j = 1...m,$ one obtain

$$\overline{\theta} * \overline{g} = \overline{\theta} * \mathbf{S} * \overline{\psi} = \overline{F} * \overline{\psi}$$

Need to note that, vector $\overline{\psi}$ is determined only with future payoff matrix **S** and with vector $\overline{g}$ and does not depend on a set of probable future events. By knowing the vector $\overline{\psi}$, it is possible to find out the price of any security, just by scalar multiplying of payoff vector $\overline{F}$ to $\overline{\psi}$. The vector $\overline{\psi}$ has a rather simple financial mean. As mentioned earlier, one of the assets is riskless bond and if, for instance first asset be such, then the first row of matrix ***S*** consist of numbers (1+*r*) and first coordinate of vector $\overline{g}$ is 1. Therefore one have,

$$\sum_{j=1}^{m} \psi_j = \frac{1}{1+r}.$$

Let us pass vector $\overline{p} = (p_1, ..., p_m)$:

$$p_j = (1+r)\psi_j.$$

By virtue the component positiveness of vector $\overline{\psi}$, all p_j components will be between 0 and 1, and the sum will be equal to one. Let *F* be a random variable, taking values $F_1, F_2, \ldots, F_m$ with probabilities $p_1, p_2, \ldots, p_m$, respectively, then equation $\overline{\theta} * \overline{g} = \overline{\theta} * \mathbf{S} * \overline{\psi} = \overline{F} * \overline{\psi}$, one can rewrite as follows:

$$F_0 = \overline{\theta} * \overline{g} = \frac{1}{1+r}(F_1 * p_1 + \ldots + F_m p_m) = \frac{1}{1+r} * E[F]$$

This means that the present price of any bond is equal to the discounted mathematical expectation of payoff with respect to probability $\overline{p} = (p_1, \ldots, p_m)$ which is called the risk neutral probability. This result is an example of an important general principle in option pricing known as risk-neutral valuation. The principle states that it is valid to assume the world is risk neutral when pricing options.

When exist vector $\overline{\psi}$ and with it probability p? The answer to this question gives, a well known theorem:

Theorem 2.1.1 *In a single period model with final number states were exists price state vectors if and only if no arbitrage opportunity exists.*

Theorem 2.1.2 *Let in a single period model with final state numbers, no arbitrage opportunity exists. Then a single state price vector exists if and only if the market is full.*

For completeness recall the conditions of no arbitrage opportunity in well known model with two states and two financial assets – riskless bond and risky stock (see Cox, Ross & Rubinstein (1979), Rendleman & Bartter (1979), Hull(2011), Gibson(1991), Goarder(1998), Pascucci (2011), Black & Scholes (1973), Georgiadis (2011)).

Random walk, particularly discrete random walk, is a natural tool for valuing options in binomial models. The binomial model is based on two key ideas. First is modelling continuous random walk by discrete random walk as follows: