λ-MODEL WITH POTTS COMPETING INTERACTIONS ON CAYLEY TREE OF ORDER TWO: GROUND STATES AND PHASE TRANSITIONS

BY

MOHD HAKIM BIN JAMIL

A thesis submitted in fulfilment of the requirement for the degree of Doctor of Philosophy (Computational and Thereotical Sciences)

Kulliyyah of Science
International Islamic University Malaysia

AUGUST 2021

ABSTRACT

In this research, we consider the λ-model with and without competing interaction on Cayley tree of order two. Description of ground states becomes one of the main elements to study as phase diagram of Gibbs measure for a Hamiltonian is close to the phase diagram of isolated ground states of the Hamiltonian. For the λ-model on infinite Cayley tree, we describe the set of periodic and weakly periodic ground states corresponding normal subgroup of the Cayley tree group representation. We construct 81 different combination of configurations and classify the configurations under 10 different regions so that the configurations will achieve ground states. We describe periodic and weakly periodic ground states for the considered model by using periodic and weakly periodic configurations. For the second result of the research, we consider λ-model with competing Potts interaction on Cayley tree of order two. As explained in previous section, we describe the periodic ground states for the considered model. Note that for this model, we have 12 different regions for the configurations to achieve ground states. For some domain of interactions strength, the configuration of periodic ground states cannot be achieved. By using Kolmogorov criteria, Gibbs measures for this model was described by deriving infinite volume distribution using given finite-dimensional distributions and find the probability measures with given conditional probability. By considering translation invariant Gibbs measure, we analyse the system of equations derived and study the phase transition phenomenon by proving the existence of multiple translation-invariant solutions for the system of equations. Phase transitions occurs if there exist two or more solutions.

خلاصة البحث

في هذا البحث تمت دراسة لنموذج-λ مع التفاعل المتنافسة وبدونها على شجرة كايلي من الرتبة الثانية. ويصبح وصف الحالات الأرضية أحد من العناصر الرئيسية التي تجب دراستها حيث أن مخطط الطور لمقياس جيبس لهاميلتوني قريب من مخطط الطور للحالات الأرضية المعزولة لهاميلتوني. تم وصف مجموعة الحالات الأرضية الدورية والدورية الضعيفة المقابلة بالنسبة لنموذج-λ على شجرة كايلي اللانهائية للمجموعة الفرعية العادية لتمثيل مجموعة شجرة كايلي. وقام الباحث ببناء 81 مجموعة مختلفة من التكوينات وتصنيف التكوينات ضمن 10 مناطق مختلفة بحيث تحقق التكوينات حالات أرضية. وأوضح الحالات الأرضية الدورية والدورية الضعيفة للنموذج المدروس باستخدام تكوينات دورية ودورية ضعيفة. بالنسبة للنتيجة الثانية في هذا البحث، فتمت دراسة لنموذج-λ مع التفاعل بوتس المتنافسة على شجرة كايلي من الرتبة الثانية. وتم صف الحالات الأساسية الدورية للنموذج المدروس كما وصف في القسم الأول. لاحظ الباحث أنه بالنسبة لهذا النموذج، لدينا 12 مناطق مختلفة للتكوينات لتحقيق حالات أرضية. وتكوين الحالات الأرضية الدورية لا يمكن أن يتحقق لبعض مجالات قوة التفاعلات. وتم وصف مقاييس جيبس لهذا النموذج من خلال اشتقاق توزيع الحجم اللانهائي باستخدام توزيعات ذات أبعاد محدودة وإيجاد مقاييس الاحتمالية مع الاحتمال الشرطي المحدد باستخدام معايير Kolmogorov. وقام الباحث بتحليل نظام المعادلات المشتقة ودراسة ظاهرة انتقال الطور عن طريق إثبات وجود حلول متعددة للترجمة الثابتة لنظام المعادلات من خلال النظر في مقاييس جيبس الثابت للترجمة. تحدث انتقالات الطور إذا كان هناك حلان أو أكثر.

APPROVAL PAGE

The thesis of Mohd Hakim Bin Jamil been approved by the following:

____
Pah Chin Hee
Supervisor

____
Farrukh Mukhamedov
Co-supervisor

____
Abdumalik Rakhimov
Internal Examiner

____
Mohd. Salmi Md Noorani
External Examiner

____
Anvarjon Ahmedov
External Examiner

____
Md. Zaidul Islam Sarker
Chairman

DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

Mohd Hakim Bin Jamil

Signature .... Date 3 September 2021 ....

INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA

DECLARATION OF COPYRIGHT AND AFFIRMATION OF FAIR USE OF UNPUBLISHED RESEARCH

λ-MODEL WITH POTTS COMPETING INTERACTIONS ON CAYLEY TREE OF ORDER TWO: GROUND STATES AND PHASE TRANSITIONS

I declare that the copyright holders of this dissertation are jointly owned by the student and IIUM.

No part of this unpublished research may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise without prior written permission of the copyright holder except as provided below

1. Any material contained in or derived from this unpublished research may be used by others in their writing with due acknowledgement.

2. IIUM or its library will have the right to make and transmit copies (print or electronic) for institutional and academic purposes.

3. The IIUM library will have the right to make, store in a retrieved system and supply copies of this unpublished research if requested by other universities and research libraries.

By signing this form, I acknowledged that I have read and understand the IIUM Intellectual Property Right and Commercialization policy.

Affirmed by Mohd Hakim Bin Jamil

……………………………… 3 September 2021 ………………………………

Signature Date

ACKNOWLEDGEMENTS

First and for most, thank to Allah, The most Beneficent, The Most Merciful, for giving me opportunity and great health for me to complete my degree in PhD. In addition, may blessing and peace be upon Prophet Muhammad who was sent for mercy to the world.

There are many people who have helped with the completion of my PhD thesis directly and indirectly. I sincerely thankful to my supervisor, Asst. Prof. Dr. Pah Chin Hee, his help and patience in advising, assisting, and encouraging me while finishing this project. Also, I would like to thank my co-supervisor, Prof. Dr. Farrukh Mukhamedov for always guiding me in solving all problem related to my research. Special thanks to Dr Muzaffar Rahmatullaev for guiding me on doing the thing that are not familiar for me and help me to understand it very well. Not to forget about how thankful he is on congratulate me for any special occasions happened to me.

I am thankful to the examiners of my thesis, Dr Anvarjhon, Dr Mohd Salmi Md Noorani and Dr Abdumalik Rakhimov, for their thorough reading of my thesis. Their unusually careful readings and thoughtful suggestions have resulted in improvements in this thesis. Moreover, I am also extremely acknowledge the financial support from Pah Chin Hee by RIGS grant scheme for one year of my study in completing this research.

On top of that, I am grateful especially to my mother, Faridah Binti Ma'amor, my late father, Jamil bin Ismail, my late grandmother, Juriah Binti Diron and my late grandfather, Ma'amor Bin Tuidin for keep supporting me by giving moral support each time I'm having internal and external problems during completing my PhD study. Eventhough your not in this world anymore, all of you are in my mind forever and ever. Moreover, thanks to my wife, Maisarah Binti Abdullah for all the support. May Allah give rewards for you, in this world and hereafter. Lastly, special thanks to all my friends, PGKOSSS community for all the support and activities that bring enjoyment in my life. May Allah ease all of you in completing your studies.

TABLE OF CONTENTS

Abstract ii
Abstract in Arabic iii
Approval Page iv
Declaration v
Copyright Page vi
Acknowledgements vii
Table of Contents viii
List of Figures x
List of Symbols xii
CHAPTER ONE 1
INTRODUCTION 1
1.1 Introduction 1
1.2 Literature Review 2
1.3 Research Objectives 5
1.4 Overview of the Thesis 6
CHAPTER TWO 8
PRELIMINARIES 8
2.1 Cayley Tree 8
2.2 λ-model 11
2.3 Ground States 13
2.4 Gibbs Measure for λ-Model 14
2.5 Phase Transitions 15
CHAPTER THREE 16
PERIODIC AND WEEKLY PERIODIC GROUND STATES OF λ-MODEL ON CAYLEY TREE OF ORDER TWO 16
3.1 Introduction 16
3.2 Hamiltonian 16
3.3 Periodic Ground States 20
3.4 Weakly Periodic Ground States Index Two 26
CHAPTER FOUR 33
λ-MODEL WITH COMPETING INTERACTIONS ON THE CAYLEY THREE OF ORDER TWO 33
4.1 Introduction 33
4.2 Periodic Ground States for λ–Model with Competing Potts Intractions on Cayley Tree of Order Two 34
4.3 Gibbs Measure of the λ-Vannimenus model 52
4.4 A Solution to the System of Equations (4.34) 63
CHAPTER FIVE 67
CONCLUSION AND FUTURE WORKS 67
5.1 Summary of Research 67
5.2 Suggestion for Future Research 68
REFERENCES 69
APPENDIX A 72

APPENDIX B **73**
APPENDIX C **74**

LIST OF FIGURES

Figure No.		Page No.
2.1	Cayley Tree of order k, Γ^2	9
2.2	Cayley Tree Γ^2 and Elements of the Group Representation of Vertices	9
2.3	Figure for the Ball, b_x	14
3.1	Figure for the Ball, b_x	20
3.2	Configurations for C_3	22
3.3	Cayley Tree After Maping Configuration (3.11)	22
3.4	Configurations for C_7	23
3.5	Cayley Tree After Maping Configuration (3.12)	24
3.6	Configurations for C_2	25
3.7	Cayley Tree After Maping Configuration (3.13)	26
4.1	A Ball, b	38
4.2	Configurations for D_1	38
4.3	τ_1-Construction for D_2 Constructed with Spin Values {1,2}	39
4.4	τ_1-Construction for D_2 Constructed with Spin Values {2,3}	39
4.5	Example for Cayley Tree from Figure 4.3	41
4.6	Reduced Cayley Tree Constructed from Configuration in D_2	43
4.7	Configurations for D_3	43
4.8	Reduced Cayley Tree Constructed from Configuration in D_3	45
4.9	Reduced Cayley Tree Constructed from Configuration in D_5	47

4.10 Configurations for D_6 48

B.1 Cayley tree Γ^2 and Elements of the Group Representation of Vertices 73

C.1 Cayley Tree Γ^2 Constructed Under Quotient Group $G_k / H_A = \{H_A, G_k \backslash H_A$ 74

LIST OF SYMBOLS

Symbol	
Λ	Finite volume
$d(x,y)$	Distance between x and y
V	Set of Vertices
$\sigma(x)$	Spin variable of x
$\Omega(\Lambda)$	Set of all configurations on Λ
T	Temperature
T^*	Critical Temperature
Γ^k	Cayley tree of order k
$<x,y>$	Nearest-neighbor
$>x,y<$	Next Nearest-neighbor
$H(\sigma)$	Hamiltonian of configurations
W_n	Set of all vertices at n^{th} level
$\mathbb{R}$	Real number
λ	Lambda function
b	Ball
M	Set of balls
S_x	Successor of x
C_{b_x}	Canter of ball
$C_{b_\downarrow}$	Vertex positioned below the center of a ball
$x_\downarrow$	Spin value below from center of a ball

φ_b	Configurations in a ball
$U(\sigma_b)$	Energy of configurations in a ball
$H_{\{a_1\}}$	Normal subgroup with consist of even number of a_1 in text of elements in vertices x.
$\sigma(C_{b_\downarrow})$	Spin for the vertex positioned below the center of a ball
$\sigma(C_b)$	Spin for the center of a ball
x_b	Vertex x above the center of a ball
$\sigma(x_b)$	Spin for the vertex x above the center of a ball

CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION

In statistical mechanics, scholars are interested with the average properties of a mechanical system. Some examples of mechanical system are water in a bottle, the atmosphere inside a room and the number of atoms in a magnet. This kind of system are normally made up of a large number of components, usually molecules. However, in order to study the component of the system, the observer has limited capability to consider all of the component. As far as researchers are concerned, to consider an actual system with many (abstractly speaking, infinitely many) degrees of freedom, we cannot consider all properties of such a system into account without putting some exceptions. Consciously, researchers had no choice and neglected certain features. One of the way to overcome this restrictions, researchers start to specify a few average quantity of the system such as its density, pressure or temperature. Throughout this specifications, the main objective of statistical mechanics is by given only a knowledge of the microscopic interactions between the components, researchers intend to predict and verify the relation with the observable macroscopic properties of the components. This problem can be well-explained by the mathematical framework. For an example, microscopic systems behavior for the freezing and boiling process of water. This kind of process can be well-described through the simplified model of structure constructed from the interactions of between atoms contained in the object, by this example scenario is water itself. In order to study the problem related to the equilibrium statistical mechanics, the choice of the Hamiltonian for the systems of interacting particles are important (Baxter, 1982). On the other hand, Georgii (1988) stated that the main goal of equilibrium statistical mechanics is to described all limit Gibbs distributions if given a Hamiltonian of the system. This problem can be completely solved only in some relatively simple cases. In the given certain condition, if there are only binary interactions in the system considered, then the problem of describing the limit Gibbs distributions is significantly simplified. Phase diagram of isolated (stable) ground states of the Hamiltonian is close to the the phase diagram of Gibbs measures for a Hamiltonian. A periodic ground states corresponds to a periodic Gibbs measure (Sinai (1982); Minlos (2000)) at low temperatures. There are many mathematical model has been constructed by the researchers in order to understand this system, e.g. Ising model, Potts Model, and many more.

1.2 LITERATURE REVIEW

The main objective of statistical mechanics studies is to describe the relation between observable macroscopic properties of the components given by only knowledge of microscopic interactions between components. As known by researchers, Ising model is the simplest model in statistical mechanics which has wide practical applications and theoretical interest (Georgii (1988); U. A. Rozikov (2013)). There are several papers which are related to the study for description of this set for the Ising model on a Cayley tree. The exact solution of an Ising model with competing interactions on a Cayley tree was studied by Ganikhodjaev et al. (2003). However, this Ising model has lacking in completing the result about all Gibbs measures. Early in 1982, Vannimenus introduced a method on describing phase diagram for Ising model with competing interactions. This phase diagram constructed from numerical value of system of equation. For this problem, Ising model with next nearest-neighbor interactions on the Cayley tree was considered. This method of describing the phase diagram lead to the increasing of interest for the researchers. Ganikhodjaev et al. (2011) described the phase diagram of an Ising system with competing binary, prolonged ternary and next-nearest interactions on a Cayley tree. Ground states related were also been studied for this ising model. Ising model on a general Cayley tree with competing interactions of next-nearest-neighbour of two types; prolonged and one-level K-tuple interactions was studied (Ganikhodjaev & Zakaria, 2011). Ganikhodjaev & Rodzhan (2014) studied the phase diagram for the Ising model that define on Cayley tree-like lattice; pentagonal chandelier, with competing one-level pentagon interactions. In this research works, pentagonal chandelier can be viewed as another geometrical representation on Cayley tree of order 5. The phase diagram was investigated and it showed the appearance of several features and modulated phase arising from frustration effects introduced by one-level pentagon interaction for several ranges of the competing parameters. Later on, new weakly periodic (non-periodic) Gibbs measures corresponding to normal subgroups of indices two in the group representation of the Cayley tree of order five and six was studied by Ganikhodjaev et al. (2017).

On the Other hand, Potts (1952) introduced Potts model as generalization of the Ising model. This Potts model encompasses a number of problems in statistical physics (see also (Wu, 1982)) due to its spin value can be consider more than two components. By these facts, this Potts model grows interest among the researchers. The critical behavior of the two-dimensional, q-state Potts model, using finite-size scalling and transfer matrix methods was investigated (Nightingale & Schick, 1952). This model simply described the special class of statistical mechanics system. Later on, the Potts model with competing interacions on the Cayley was investigated. This model is more complex and has rich structure of ground states. Ganikhodjaev (1990) described the pure phase of the ferromagnet Potts with three states of Betthe lattice of order two. Later on, Botirov & Rozikov (2007) considered Potts model with three spin values and with competing interaction of radius $k=2$. In this paper, they described the ground states and proved non uniqueness of Gibs measures for the model by using contour method. Ganikhodjaev et al. (2007) studied the Potts model on a Cayley tree in the presence of competing two binary interactions and magnetic field. This paper solved for the phase transitions problem and the critical curve for the phenomenon of phase transitions was found. As the continuation of the research by Ganikhodjaev (1990), phase diagram of the three-states Potts model with competing nearest neighbor and next nearest-neighbor interactions on a Cayley tree has benn numerically obtained (Ganikhodjaev et al., 2008). The two-dimensial q-component Potts model is equivalent to a staggered ice-type model. It is deduced that the model has first-order phase transitions for $q > 4$, and a higher-order transition for $q \leq 4$. The free energy and latent heat at the transition were calculated (Baxter, 2011). Periodic Gibbs measure was studied for the Potts model on the Cayley tree (Rozikov & Khakimov, 2013). In this research, the condition for the considered model with nonzero external fields admits periodic Gibbs measure was found.

It s natural to consider more complicated models that the Potts one, so called λ-model. This λ-model was first introduced by Rozikov (1998). In this paper, limit Gibbs measures for the λ-model on the Bethe lattice was described by considering spin values $\{-1,+1\}$. Motivated by this research potential, Mukhamedov (2004) continued the study by investigating the factor associated with the unordered phase of λ-model on a Cayley tree. Phase transition phenomenon are not considered in this research work. On the latest research works, Mukhamedov et al. (2017) described the ground states for the λ-model on Cayley tree of order two. As the continuation of the studies, phase transition phenomenon was studied for the considered model (Mukhamedov et al., 2018). As far as comcern, there are not so many research paper realted to the λ-model on Cayley tree.

The phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of the Hamiltonian. A periodic ground states corresponds to a periodic Gibbs measure (Sinai (1982); Minlos (2000)) at low temperatures. This is the main motivation why researchers intend to described the Gibbs measure of lattice models. Weakly periodic ground states of Ising model with competing interactions on cayley tree was described (Rozikov & Rahmatullaev, 2008). For the past few years, ground states related problem also were studies for this Potts model. Rahmatullaev (2013) described weakly periodic Gibbs measures and ground states for the Potts model with competing interactions on the Cayley tree. Some explicit formula of the free energies and entropies (according to vector-valued boundary conditions (BCs) are obtained for the Potts model on the Cayley tree which are corresponding to weakly periodic Gibbs measures. Later on, periodic and weakly periodic ground states for the Potts model with competing interactions on the Cayley tree was studied (Rahmatullaev, 2016).

1.3 RESEARCH OBJECTIVES

As mentioned earlier, the description of ground states become one of important feature as a gorund states corresponds to a Gibbs measures at low temoeratures. For this thesis, we considered λ –model on Cayley tree of order two. There are several objectives were set up for this research. For the first part of the research, we consider λ– model on the Cayley tree of order two. For this considered model on Cayley tree of order 2 , we intend

i. to describe the periodic ground states for the λ-model on Cayley tree;.

ii. to describe the weakly periodic ground states.

For the second part of this project, we consider λ-model with Potts competing interaction on cayley tree of order two. For the λ-model with Potts competing interactions on Cayley tree, we intend

iii. to describe the periodic ground states;

iv. to construct the Gibbs measures for the λ-model;

v. to analyze the system of equations for the considered model and prove the existence of phase transition phenomenon.

1.4 OVERVIEW OF THE THESIS

This thesis consist of three parts.

In chapter 2, we will discuss about the basic notion and setting that had been used to complete this research. We provide the definition of the Cayley tree, Gibbs measures, ground states, the model considered for this research and phase transition phenomenon. As mentioned, there are two type of models we considered for this research. In order to complete the first and second parts of the objectives, we consider λ-model on Cayley tree of order two. As for the third, fourth and fifth objectives of this research, we consider λ-model with Potts competing interactions on Cayley tree of order two.

In chapter 3, we focus on completing the first and second objectives of my research. Note that this research are coninuation from my previous studies during my master study. In this chapter, λ-model on cayley tree was introduced. Then, the periodic and weakly periodic ground states for the considered model are described. All proof of the existence of this ground states, either periodic or weakly periodic cases wre provided in this chapter.

In chapter 4, we considered the λ-model with competing Potts interactions on Cayley tree of order two. Noted that the model has additional Potts interaction compare to previous model in chapter 3. In this chapter, we will cober the nxt three objectives for my research. For the first part of this chapter, periodic ground states was considered for the model. For the second part, we will construct the Gibs measures for the considered model. For this part, we proved for the consistency and efficiency of the Gibbs measures constructed. For the third part, we proved the existence of phase transitions phenomenon for the model considered on cayley tree.

CHAPTER TWO

PRELIMINARIES

2.1 CAYLEY TREE

In this chapter, we will introduce the notion of Cayley tree, lambda model, Gibbs measure, ground state and phase transition. We also discuss the group structure of Cayley tree that associate with the vertex of the Cayley tree.

Let $\Gamma^k = (V, L)$ be a Cayley tree of order k, i.e, an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k. For an arbitrary vertex $x_0 \in V$, we put

$$W_n = \{x \in V \mid d(x_0, x) = n\}, V_n = \bigcup_{m=0}^{n} W_m, L_n = \{l = < x, y > \in L \mid x, y \in V_n\}.$$

where $d(x, y)$ is the distance between x and y in the Cayley tree, i.e., the number of edges of the path between x and y.

Let G_k denote the free product of $k+1$ cyclic groups $\{e, a_i\}$ of order 2 with generators $a_1, a_2, \ldots a_{k+1}, a_i^2 = e.$

There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order k and the group G_k (Rozikov & Khakimov, 2013).

For the sake of completeness, we show how to construct this correspondence. We choose an arbitrary vertex $x_0 \in V$ and associate it with the identity element e of the group G_k. Since we may assume that the graph under consideration is planar, we associate each neighbor of x_0

(i.e., e) with a single generator $a_i, i = 1, 2, ..., k+1,$ where the order corresponds to the positive direction, see Figure 2.2.

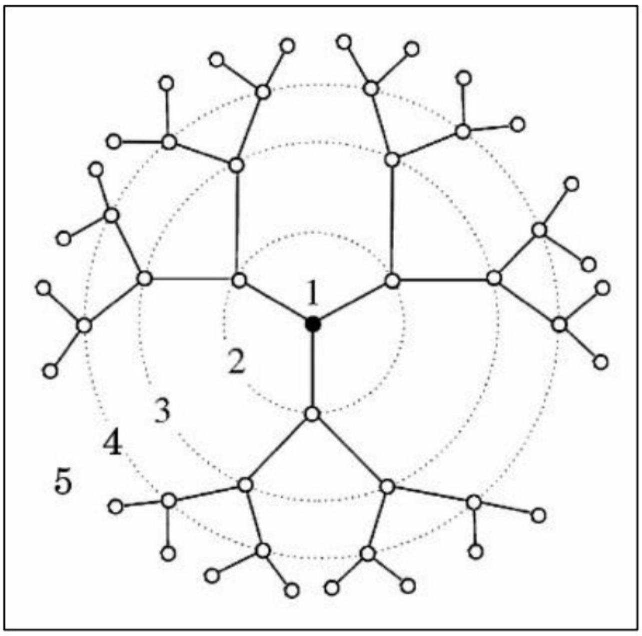

Figure 2.1: The Cayley tree of order 2, Γ^{2}

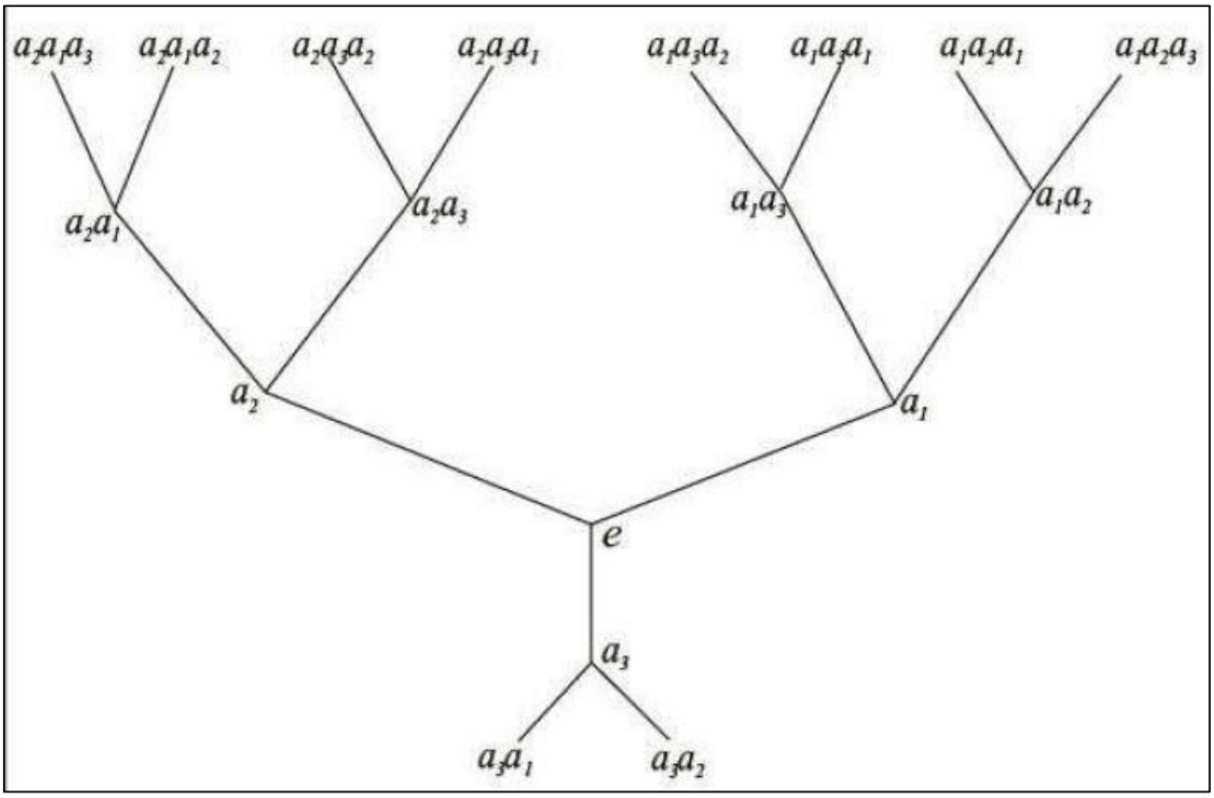

Figure 2.2: The Cayley tree Γ^{2} and Elements of the Groups Representation of vertices

For every neighbor of a_i, we introduce words of the form $a_i a_j$. Since one of the neighbors of a_i is e, we put $a_i a_j = e$. The remaining neighbors of a_i are labeled according to the above order. For every neighbor of $a_i a_j$, we introduce words of length 3 in a similar way. Since one of the neighbors of $a_i a_j$ is a_i, we put $a_i a_j a_j = a_i$ ai . The remaining neighbors of $a_i a_j$ are labeled by words of the form $a_i a_j a_l$, where $i, j, l = 1, 2, ..., k+1$, according to the above procedure. This agrees with the previous stage because $a_i a_j a_j = a_i a_j^2 = a_i$. Continuing this process, we obtain a one-to-one correspondence between the vertex set of the Cayley tree Γ^k and the group G_k.

The representation constructed above is said to be right because, for all adjacent vertices x and y and the corresponding elements $g, h \in G_k$, we have either $g = ha_i$ or $h = ga_j$ for suitable i and j. The definition of the *left* representation is similar.

For the group G_k (or the corresponding Cayley tree), we consider the left (right) shifts. For $g \in G_k$, we put

$$T_g(h) = gh(T_g(h) = hg)$$

for all $g \in G_k$. The group of all left (right) shifts on G_k is isomorphic to the group G_k.

Each transformation S on the group G_k induces a transformation S on the vertex set V of the Cayley tree Γ^k (U. A. Rozikov, 2013). From now on, we identify V with G_k.

Theorem 2.1.1 [U. A. Rozikov (2013)]: *The group of left (right) shifts on the right (left) representation of the Cayley tree is the group of translations.*

By the group of translations we mean the automorphism group of the Cayley tree regarded as a graph. Recall that a mapping ψ on the vertex set of a graph G is called an automorphism of G if ψ preserves the adjacency relation, i.e., the images $\psi(u)$ and $\psi(v)$ of vertices u and v are adjacent if and only if u and v are adjacent.

For each $x \in G_k$, let $S(x)$ denote the set of immediate successor of x, i.e., if $x \in W_n$, then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k :< x, y > \in L\}$. The set $S_1(x) \setminus S(x)$ is a singleton. Let $x_\downarrow$ denote the (unique) element of this set.

Spin values are introduced as the value at the vertex on a Cayley tree. Assume that spin takes its values in the set $\Phi = \{1, 2, ..., q\}$. By a configuration on V denoted by σ we mean a function taking $\sigma : x \in V \to \sigma(x) \in \Phi$. The set of all configurations coincides with the set $\Omega = \Phi^V$.

2.2 λ-MODEL

Lattice model is a model defined on a lattice, e.g. square lattice or Cayley tree. Researchers can study the phase transition phenomenon by describing all limits Gibbs measures corresponding to the Hamiltonian defined on the lattice model. Ising model was first introduced by Ising (1925) that consider spin values $\{-1, +1\}$. For this Ising model, the interaction between two vertices are introduced as multiplication of the spin value of the vertices. This model was considered as the simplest model that attract wide theoretical interest. Later on, Potts model was introduced as generalization of the Ising model, has more than two components. λ-model was