SOME CLASSES OF 2-PARTITION GEOMETRIC QUADRATIC STOCHASTIC OPERATOR ON COUNTABLE STATE SPACE AND ITS REGULARITY

BY

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A thesis submitted in fulfilment of the requirement for the degree of Master of Science (Computational and Theoretical Sciences)

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ABSTRACT

This research is designed to define and construct some classes of Geometric quadratic stochastic operator defined on the countable state space generated by 2-partition of finite and infinite points, and to investigate their trajectory behaviour. These operators can be reinterpreted in terms of evolutionary operator of free population with arbitrary initial measure. It is shown that such operators are regular transformations through the convergence of the trajectory behaviour to a fixed point in the open interval $(0,1)$ which indicates the existence of the strong limit of the sequence of trajectories.

خلاصة البحث

تم تصميم هذا البحث لتحديد وبناء بعض من فئات مشغلات مؤشر ستوكاستيك من الدرجة الثانية المحددة على مساحة الحالة القابلة للعد والتي تم إنشاؤها بواسطة قسمين من النقاط المحدودة وغير المحدودة وأيضا للتحقق من سلوكهما المساري. بالإمكان إعادة تفسير هذه العوامل بناء على المشغل التطوري للتعداد الحر بالحساب الأولي الافتراضي. اتضح أن هذا المشغل عبارة عن تحول منتظم من خلال تقارب السلوك المساري إلى نقطة ثابتة في الفترة المفتوحة (0، 1) مما يدل على وجود حد قوي لتسلسل المسارات.

APPROVAL PAGE

I certify that I have supervised and read this study and that in my opinion, it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science (Computational and Theoretical Sciences).

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DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

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LIST OF SYMBOLS

V	Operator
X	State space
F	σ-algebra
(X,F)	Measurable space
$S(X,F)$	Set of all probability measure space on (X,F)
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$P_{ij,k}$	Coefficient of quadratic stochastic operator
$Fix(V)$	Set of fixed point of operator V
ξ	Partition

CHAPTER ONE

INTRODUCTION

1.1 RESEARCH BACKGROUND

Quadratic stochastic operator (QSO) was originally established by Bernstein through his advanced work on population genetics. Bernstein's work was based on the three main parts of the problems arise in the evolution theory initiated by Charles Darwin in 1859 as Bernstein explained his aim to study the interrelation between the Mendelian law of crossing and the Galton law of regression. He stated that his main axiom was *the Darwin law of stationarity*, which, as he added, was as important in the heredity as the inertia law was in mechanics. Later, he introduced and described his work on the mathematical investigation on population genetics as the solution for the problems discovered in the theory of evolution (Bernstein, 1924). Bernstein's work has triggered the idea of heredity in QSO. The theory of QSO was frequently appeared and considered a significant source of analysis for the investigation of dynamical properties and modeling in different domains such as biology (Akin, 1993; Bernstein, 1924; N. N. Ganikhodzhaev & Zanin, 2004), physics (R. N. Ganikhodjaev, 1993), game theory (R. N. Ganikhodjaev, 1994; R. Ganikhodzhaev, Mukhamedov, & Rozikov, 2011; Hofbauer & Sigmund, 1998; Jenks, 1969), mathematics (Lyubich, 1992; Ulam, 1964), etc..

QSO is generally used to present the time evolution of differing species in biology, thus it is also known as *evolutionary operator*. In order to comprehend how such kind of operators arise in population genetics, let us consider biological population, that is, a group of organisms closed with respect to reproduction.

Consider a population consisting of m species (or traits). We denote a set of all species (traits) by $I = \{1, 2, ..., m\}$. Let $\mathbf{x}^{(0)} = \left(x_1^{(0)}, ..., x_m^{(0)}\right)$ be a probability distribution of species at an initial state and $P_{ij,k}$ be a probability that individuals in the i^{th} and j^{th} species (traits) interbreed to produce an individual from k^{th} species (trait). Then a probability distribution $\mathbf{x}^{(1)} = \left(x_1^{(1)}, ..., x_m^{(1)}\right)$ of the species (traits) in the first generation can be found as a total probability,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)},\ k = 1, ..., m, \tag{1.1.1}$$

The association $\mathbf{x}^{(0)} \to \mathbf{x}^{(1)}$ defines a mapping V that is called a quadratic stochastic operator (QSO). The above association means that the population evolves by starting from an arbitrary state $\mathbf{x}^{(0)}$, then passing to the state $\mathbf{x}^{(1)} = V\left(\mathbf{x}^{(0)}\right)$ which indicates the probability of the first generation, then to the state, $\mathbf{x}^{(2)} = V\left(\mathbf{x}^{(1)}\right) = V\left(V\left(\mathbf{x}^{(0)}\right)\right) = V^2\left(\mathbf{x}^{(0)}\right)$, the second generation, and so on. Therefore, the evolution states of the population system can be described by the following discrete dynamical system,

$$\mathbf{x}^{(0)}, \mathbf{x}^{(1)} = V\left(\mathbf{x}^{(0)}\right), \mathbf{x}^{(2)} = V^2\left(\mathbf{x}^{(0)}\right), \mathbf{x}^{(3)} = V^3\left(\mathbf{x}^{(0)}\right), ..., \tag{1.1.2}$$

where $V^n(x) = \underbrace{V\left(V\left(...V(x)\right)...\right)}_{n}$ denotes the n times iteration of V to x.

In other words, a distribution of the next generation can be described by QSO if the distribution of the current generation is given. We should emphasize that the mapping V defined by (1.1.1) is a nonlinear (quadratic) operator and it is higher-dimensional if $m \geq 3$. Higher dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood, for example, pendulum and solar system in mechanics, and evolution in biology.

The main problem for a given dynamical system (1.1.2) is to describe the limit points of $\left\{x^{(n)}\right\}_{n=0}^{\infty}$ for arbitrary given $x^{(0)}$.

Nonlinear operator theory falls within the general area of nonlinear functional analysis, an area which has been receiving a lot of interest in recent years. One of the central problems in the nonlinear operator theory is to study the asymptotical behavior of nonlinear operators. The problem in nonlinear operator is still not depleted even in the class of QSO which is perceived as the simplest nonlinear operators. To study this problem, several classes of QSO were constructed and investigated in many publications.

In this thesis, we study some classes of Geometric quadratic stochastic operator on a countable state space. We first define and construct a class of Geometric QSO generated by 2-partition of several points on a countable state space and later we consider 2-partition of infinite points. Further, we investigate the limit behaviour of trajectories of the Geometric QSO. Next, the dynamical systems of the constructed QSO are investigated in order to show their regularity.

1.2 RESEARCH OBJECTIVES

In this thesis, our main objectives of the study are:

1. To define and study the Geometric quadratic stochastic operators defined on a countable state space.
2. To construct the Geometric quadratic stochastic operators generated by 2-partition of finite points defined on a countable state space.

3. To construct the Geometric quadratic stochastic operators generated by 2-partition of infinite points defined on a countable state space.
4. To investigate and describe the trajectories of the Geometric quadratic stochastic operators generated by 2-partition of finite and infinite points and their regularity.

1.3 SCOPE OF RESEARCH

This research focuses on the construction of some classes of Geometric quadratic stochastic operators generated by 2-partition of finite and infinite points and the study of their trajectory behaviour defined on a countable state space.

1.4 THESIS ORGANISATION

This thesis is organized into six chapters. Chapter 1 serves as the introduction to the whole thesis. In Chapter 1, we introduce the concept of quadratic stochastic operator as the background of the research. Chapter 1 also presents the objectives and scope of this research.

Next, Chapter 2 presents the literature review of this research. The first two sections deal with set theory and measure theory where some notions and notations that we use are given. The concepts of quadratic stochastic operator, regularity of quadratic stochastic operator, and Geometric quadratic stochastic operator are discussed throughout the remainder of this chapter.

In Chapter 3, we give a thorough explanation of the construction of Geometric QSO generated by 2-partition of singleton and two points on a countable state space. This chapter also investigates the regularity of such Geometric QSO. In the first section, we explain the construction of Geometric QSO generated by 2-partition of singleton and followed by the investigation on the regularity of such operator numerically. Then, we provide the analytical proof through the concept of fixed point. We break down the discussion on Geometric QSO generated by 2-partition of two points into three sub-sections, i.e. consecutive and nonconsecutive two points, and finally arbitrary two points as the study of those parts is indispensable for a profound understanding on the result.

Then, Chapter 4 provides the result for the study of Geometric QSO generated by 2-partition of finite points. It may serve as the presentation of the extended case discussed in the previous chapter. Through this chapter, the regularity of the constructed Geometric QSO is studied and discussed based on their trajectory behaviour.

Chapter 5 is the final chapter that constitutes the heart of this thesis. Section 5.2 gives a detailed discussion on Geometric QSO generated by 2-partition of modulus 3. The notion of "modulus" is used to introduce a set of infinite points on a countable state space. Later, the generalized case of infinite points is presented in Section 5.3.

Lastly, Chapter 6 gives a summary of this research. Some suggestions for future research are also provided in this chapter.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter presents the literature obtained from journal papers and related books. The basic concept of set theory, measure theory, quadratic stochastic operator, regularity of quadratic stochastic operator, and Geometric quadratic stochastic operator will be presented. These concepts are used and discussed throughout this thesis.

2.2 SET THEORY

Some basic definitions and properties on set theory are taken from (Bartle & Sherbert, 1999; McDonald & Weiss, 2013).

A set is a collection of elements. If A is a set and x is an element of A, then we write $x \in A$, meanwhile $x \notin A$ means that x is not an element of A. The symbol $\varnothing$ denotes the *empty set*, a set containing no elements.

Let A and B be sets. If every element of A is an element of B, then A is said to be a *subset* of B, denoted $A \subset B$ or $B \supset A$. Two sets A and B are *equal* if they contain the same elements, where $A \subset B$ and $B \supset A$. If $A \subset B$ but $B \not\subset A$, then we say that A is a *proper subset* of B.

We will now mention some fundamental operations on set, namely, complement, intersection, and union as we will use some of the concepts throughout this thesis. In what follows, we will assume that all sets under consideration are subsets of some fixed set X, generally referred to the universal set. The set of all subsets of X is called the power set of X denoted by $P(X)$. Thus, $A \subset X$ if and only if $A \in P(X)$.

Definition 2.2.1 Let A and B be sets.

(i) The union of sets A and B is the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\} .$$

(ii) The intersection of the sets A and B is the set

$$A \cap B := \{x : x \in A \text{ and } x \in B\} .$$

(iii) The complement of B relative to A is the set

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$

Consider all possible basic outcomes of a statistical experiment be a set X, then the set X is called a *state space* of all the outcomes. The elements of set X are called outcomes.

A set X is finite with cardinality $n \in \mathbb{N}$ for some natural number, if there exists a one-to-one and onto function $f : \{1,...,n\} \to X$; otherwise X is infinite.

In this thesis, we consider a countable set X on infinite space.

Definition 2.2.2 Any set X which is equivalent that is one-to-one correspondence to the set of all natural numbers is called a *countable set of* X .

The elements of countable set X can be numerated by the natural number, i.e. $X = \{x_1, x_2, ..., x_n, ...\}$.

Definition 2.2.3 A set X which is equivalent that is one-to-one correspondence to $[0,1]$ is called *continuous set* X .

It is known that the continuous set X is uncountable set.

Note that, in this thesis, we shall define two sets to represent two measurable partitions on a countable state space $X = \{0,1,2,...\}$ which later employed on the Cartesian plane.

Definition 2.2.4 If A and B are nonempty sets, then the Cartesian product $A \times B$ of A and B is the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$, that is,

$$A \times B := \{(a,b) : a \in A, b \in B\}.$$

Definition 2.2.5 An n-dimensional simplex in a Euclidean space denoted as $\mathbb{R}^n$ consists of $n+1$ linearly independent points $p_0, p_1, ..., p_n$ together with all line segment $a_0 p_0 + a_1 p_1 + ... + a_n p_n$ where $a_0 \geq 0$ and $a_0 + a_1 + ... + a_n = 1$.

Due to the compactness and convexity properties of the simplex, one of the main problems in QSO is to study the asymptotical behavior of its trajectories. We shall let $\mathbb{R} = (-\infty, \infty)$ be a set of real numbers, then the notion of compact set and convex set are given below.

2.2.1 Compact Set

The definitions on compact set are taken from a book by Holmgren (1994).

Definition 2.2.6 Let Y be a subset of real numbers, $Y \subset \mathbb{R}$. Then, a real number x is an accumulation point of Y if every neighborhood of x contains an element of Y which is other than x.

Definition 2.2.7 A set $Y \subset \mathbb{R}$ is *closed* if it contains all of its accumulation points.

Definition 2.2.8 A set of real numbers, $Y \subset \mathbb{R}$ is *bounded* if there exists a positive number which is larger than every element of the set. In other words, there exists a closed interval $[a,b]$ which contains the set.

Definition 2.2.9 A set of real numbers, $Y \subset \mathbb{R}$ is called a *compact set* if Y is closed and bounded.

2.2.2 Convex Set

The definitions on convex set are taken from a book by Junghenn (2015).

Definition 2.2.10 A *vector space* consists of a set X, a field of scalar F, and two functions vector addition $+: X \times X \to X$ and scalar multiplication $\cdot: F \times X \to X$, such that the following conditions are satisfied for all $x, y, z \in X$ and $\alpha, \beta \in F$:

a) $x + y = y + x$,

b) $x + (y + z) = (x + y) + z$,

c) there exists $0 \in X$ such that $x + 0 = x$,

d) there exists $-x \in X$ such that $x + (-x) = 0$,

e) $\alpha(\beta x) = (\alpha\beta)x$,

f) $\alpha(x + y) = \alpha x + \alpha y$,

g) $(\alpha + \beta)x = \alpha x + \beta x$,

h) there exists $1 \in X$ such that $1 \cdot x = x$.

Definition 2.2.11 A subset S of a vector space is said to be a convex set if for all $x, y \in S$, $0 \le \alpha \le 1$, we have $\alpha x + (1 - \alpha) y \in S$.

The above definition can be restated as, a set S is convex if for any two points x and y belonging to S, there are no points on the line between x and y that are not members of S. In other words, whenever S contains two points, it also contains the entire line segment that connects the two points.

The notion of one of the crucial concepts in this thesis, measure is discussed in the next section.

2.3 MEASURE THEORY

In this section, we provide the general definition of a measure. In mathematical analysis, a measure on a set is a systematic way to assign a number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume.

The basic definitions of algebra are presented below in order to define a measure. The definition of the algebra is taken from a book by Halmos (2014).

Let X be any set and $P(X)$ is a system containing all subsets of X, that is $P(X)=\{A : A\subset X\}$.

Definition 2.3.1 A non-empty system of sets $\Re \subset P(X)$ is called a *ring*, if for every $A, B \in \Re$ implies that $A \cup B \in \Re$ and $A \setminus B \in R$.

Definition 2.3.2 A ring $\Re \subset P(X)$ is called an *algebra* if $X \in \Re$.

Definition 2.3.3 A collection F of subsets of a set X is called *σ-algebra* if

(i) $\varnothing \in F$,

(ii) $A \in F$ implies that A' (complement of A) $\in F$,

(iii) $A_1, A_2, \ldots \in F$ then $\bigcup_{i=1}^{\infty} A_i \in F$.

Definition 2.3.4 For any topological space X, the σ-algebra F_X generated by the class of all open subsets of X is called a *Borel σ-algebra* of X. Any element of F_X is called a *Borel set*.

Note that, in this thesis, since we are investigating for countable state space, the corresponding σ-algebra, $P(X)$ is a power set of X.

Definition 2.3.5 Let $D \subset P(X)$ where $P(X)$ is a power set of X, then a system of sets D is called *semiring* if the following conditions are satisfied:

(i) $\varnothing \in D$,

(ii) $A \cap B \in D$ for any $A, B \in D$,

(iii) if A_1 and $A \in D$ such that $A_1 \subset A$, then there are finitely many mutually disjoint sets $A_2, A_3, \ldots, A_n \in D$, $(A_i \cap A_j = \varnothing, i \neq j)$ such that $A = \bigcup_{k=1}^{n} A_k$, $A_1 \cap A_i \neq \varnothing,\ i = 2, 3, \ldots, n$.

Then, a measure is defined as follows.

Definition 2.3.6 Let D be a semiring in $P(X)$, then a mapping $\mu : D \to [0, \infty)$ is called a *measure* μ on D if the following conditions are satisfied:

(i) $\mu(A) \geq 0$, for every $A \in D$,

(ii) μ is additive, that is, if $A = \bigcup_{k=1}^{\infty} A_k$, $A_1 \cap A_i \neq \varnothing$, $i \neq j$, $A_k \in D$, $k = 1, 2, \ldots$ for $A \in D$, then $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$.

If a measure μ defined on a σ-algebra F and $\mu(X) = 1$, then the measure is called probability measure.

In this thesis, we consider Geometric distribution as the measure, which is an example of discrete measure. The definition of discrete measure is shown below.

Definition 2.3.7 A probability measure μ on (X,F) is said to be discrete, if there exists a countably finite many elements $\{x_1,x_2,...,x_n,...\}\subset X$, such that $\mu(\{x_i\})=p_i$ for $i=1,...,n,...$, with $\sum_{i=1}^{\infty} p_i = 1$ Then, $\mu(X\setminus\{x_1,x_2,...,x_n,...\})=0$ and for any $A\in F$, $\mu(A)=\sum_{x_i\in A}\mu(x_i)$.

The simple discrete measure on set X is known as a Dirac measure, δ_x which is defined by

$$\delta_x(A)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A,\end{cases} \tag{2.3.1}$$

for any $A\in F$.

Definition 2.3.8 A set X with a σ-algebra F is called a *measurable space* and denoted by (X,F). If a measure μ is chosen for (X,F), then it is called a *measure space* and denoted by (X,F,μ) or (X,μ).

Throughout this thesis, let X be a countable set on a measurable space, then on this set, a Geometric distribution is considered. Therefore, we introduce and study the QSO on the countable set X, namely Geometric QSO.

In the following section, the concept of QSO on the set $S(X,F)$ of all probability measures is presented in details.

2.4 QUADRATIC STOCHASTIC OPERATOR

Stochastic refers to a randomly determined process. This term is used in various fields where stochastic or random processes are used to represent systems or phenomena that seem to change in a random way. In a probability theory and related fields, a stochastic or random process is a mathematical object usually defined as a family of random variables. Historically, the random variables were associated with or indexed by a set of numbers usually viewed as points in time, giving the interpretation of a stochastic process representing numerical values of some systems randomly changing over time.

Applications and the study of various phenomena have inspired the study of stochastic processes and operators which uses mathematical knowledge and techniques from probability, calculus, linear algebra, set theory, and topology as well as branches of mathematical analysis. The theory of stochastic processes is considered to be a crucial contribution to mathematics and it continues to be an active topic of research for both theoretical reasons and applications. Based on their mathematical properties, the study of quadratic stochastic operator falls into the theory of stochastic processes.

The idea of quadratic stochastic operators (QSO) was established through Bernstein's presentation of three main parts of the problems of the well-known evolution theory by Charles Darwin. The first part studies the processes of heredity irrespective of the influence of selection and environment. The second part is the study of the influence of all types of selection which presents itself as a mathematical development of the same principles. The third problem studies the influence of the environment on the variability of creatures. Thus, these indicated problems were then inspired him to study the interrelation between the Galton law of regression and the Mendelian law of crossing as he saw the demands on the application of mathematical methods of the theory of probability, which serves as the formal framework (Bernstein, 1924).

Later, in 1924, Bernstein developed an advanced work in his time, which is, a mathematical investigation on population genetics, involving a synthesis of Mendelian inheritance and Galtonian laws of inheritance. He started his notes by introducing the formula to obtain the probability distribution for the first generation, in which the application of the same iterative formulas then provided us with the probability distribution for any following generation. His mathematical investigation on population genetics for some cases was described and explained briefly, thus the heredity laws of those cases were established.

The following decades showed the interest of experts in other fields in considering QSO as an ample source of analysis for the study of dynamical properties and modeling in diverse fields of study, such as biology, physics, game theory, finance, economics, and mathematics. The study of QSO is still not exhausted even though it is considered as the simplest nonlinear operators. It is understood that the difficulty of the problem depends on the given cubic matrix $\left(P_{ijk}\right)_{i,j,k=1}^{m}$, where m is the dimensional simplex.

Here we can state our definition of cubic matrix as an object with three indices, P_{ijk}, where P_{ijk} are coefficients of heredity. Considering a population consisting of m species, P_{ijk} is the probability that individuals in i^{th} and j^{th} species interbred to produce and individual k, more precisely P_{ijk} is the conditional probability $P(k \mid i, j)$ that i^{th} and j^{th} species interbred successfully, then they produce an individual k.

The asymptotic behaviour of the QSO even on the small dimensional simplex is complicated. To deal with this problem, many researchers introduced certain classes of QSO and studied their behaviour. However, all the classes together would not cover the set of all QSO. Therefore, there are many classes of QSO which are not studied yet.

In R. N. Ganikhodjaev (1993), the theory of Volterra QSO was developed using the theory of Lyapunov functions and tournaments. However, in the non-Volterra case, many questions remain open and there seems to be no general theory available. In the