SUMMATION OF THE SPECTRAL EXPANSIONS ASSOCIATED WITH THE SOLVABILITY OF THE HEAT AND WAVE PROBLEMS

BY

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A thesis submitted in fulfillment of the requirement for the degree of Doctor of Philosophy (Engineering)

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FEBRUARY 2020

ABSTRACT

A relation between the theory of multiple Fourier series and partial differential equations was divulged in the beginning of the nineteenth century and it is known as spectral theory of the differential operators. The spectral theory of the differential operators is a significant part of mathematical sciences and it has applications in many branches of engineering. The development of the spectral theory of the differential operator is started since the time when Fourier studied heat conduction problem in a rod and the solution is found as a form of sin series. To adjust the obtained solution leads to study of the problems of the convergence of that series solutions and it depends on the initial and/or boundary data. Obtained series solutions of the problems may not be convergent. Then the problem of summability will occur. Regularization of the divergent series solution is accurate numerical interpretations of the solutions of the problems. In this research, to find the equiconvergence of the spectral expansions and to find the solution of the heat and wave problem regularization was required. In the first phase, we studied a special elliptic partial sum of order $2(m+1)$ of multiple Fourier series and integral in the spaces of singular distributions. We discussed the equiconvergence in summation of the Fourier series and integral of the linear continuous functional for specific conditions in the Lioville space. Therefore, we proved a precise equiconvergence relation between index of the Bochner-Riesz means of the expansions and power of the singularity of the distributions with compact support in summation associated with the elliptic operator. After that, we studied the vibration problem made of thin elastic membrane stretched tightly over a square frame. The deflection of the membrane during the motion is small compared to the size of the membrane. And for heat transfer problems the plate is made of some thermally conductive material. We discussed different types of heat transfer problems such as, steady state heat transfer problem, heat transfer insulated plate problems. Solution of wave and the heat transfer problems are subjected to the boundary conditions and initial conditions and had a form of double Fourier series. The coefficient of the Fourier series found from the initial conditions. Convergence of the corresponding Fourier series depends on smoothness or singularity of initial conditions. In our case, initial conditions were the Dirac delta function and it diverges. Thus for the solutions of the corresponding heat and wave problems some regularizations of the Fourier series solutions are required. Here, based on the singularity we considered the Reisz method of summation as regularization of the Fourier series solutions of the heat and wave problems. When we increased the order of the Reisz means, the solutions were convergence but the numerical calculations were increased. So, to minimize the calculations of the regularized Fourier series solutions, we optimized the regularization of the solutions of the plate vibration and heat transfer problems. For optimization of the regularized Fourier series solutions, we took minimum order of the Reisz means. The minimum order was $s > (N - 1)/2 - l$. After optimization, we used a numerical computing programming (MAT LAB) for the numerical solutions. Here, we found the optimization of the regularization of the series solutions at a fixed point of the plates at initial time and critical index. After critical point we achieved the good convergence.

خلاصة البحث

ان العلاقة بين نظرية سلسلة فورييه المتعددة والمعادلات التفاضلية الجزئية قد تم اكتشافها في بداية القرن التاسع عشر وقد عرفت باسم نظرية الطيف للمشغلات التفاضلية . تعتبر نظرية الطيف للمشغلات التفاضلية جزءا مهما من العلوم الرياضية ولها تطبيقات في كثير من فروع الهندسة .ان تطوير نظرية الطيف للمشغلات التفاضلية قد بدأ منذ الوقت الذي بدأ فيه فورييه بدراسة مشكلة التوصيل الحراري لقضيب وكان الحل للمشكلة بشكل سلسلة جيب بلغة حساب المثلثات. ان تعديل الحل المستحصل علية قد قاد الى دراسة مشاكل تقارب حلول السلاسل واعتمادها على بيانات اولية او حدية .ان حلول السلسلة للمشكلة قد لايكون متقاربا.وعندها ستحدث مشكلة الجمع او دمج السلاسل.ان تسوية اوتنظيم حل السلاسل المتباعدة هو تفسير عددي دقيق للحلول الخاصة بالمشكلة. في هذه الدراسة كان المطلوب هو ايجاد التقارب المتساوي لامتدادات الطيف ولايجاد تسوية حل مشكلة الحرارة والموجة. في المرحلة الاولى درسنا المجموع الجزئي البيضوي الخاص من مرتبة 2(m+1) لسلسلة فورييه المتعددة والتكامل في الفراغ للتوزيعات المفردة. تم مناقشة التقارب المتساوي في مجموع سلسلة فورييه وتكامل الدالة الخطية المستمرة لشروط خاصة في فراغ ليوفيل. وبالتالي تم اثبات العلاقة الدقيقة للتقارب المتساوي بين متوسطات مؤشر بوخنر–ريس للامتدادات وطاقة التفرد للتوزيعات مع دعم مدمج للمجموع المرتبط بالمشغل البيضوي. بعد ذلك تم دراسة مشكلة التذبذب الناتج من غشاء مرن رقيق امتد حول اطار مربع .ان انحراف الغشاء خلال الحركة كان صغيرا مقارنة بحجم الغشاء . وبالنسبة لمشكلة التوصيل الحراري كانت الصفيحة مصنوعة من مادة موصلة للحرارة .تم مناشة انواع مختلفة لمشاكل التوصيل الحراري , على سبيل المثال مشكلة الحالة المستقرة للتوصيل الحراري , مشاكل التوصيل الحراري للوحة معزولة. ان حل مشاكل الموجة وانتقال الحرارة تعتمد على شروط الحد والشروط الاولية ولها شكل مسلسلة فورييه الثنائية . تم ايجاد معاملات سلسلة فورييه اعتمادا على الشروط الاولية . ان تقارب سلسلة فورييه المعنية تعتمد على نعومة وسلاسة او فردية الشروط الاولية. في دراستنا فان الشروط الاولية كانت دالة ديراك دلتا وهي متباعدة. ولهذا فان حلول مشاكل الحرارة والموجة فان حلول سلسلة فورييه احتاجت الى تسوية . وهنا وبالنظر للفردية فاننا اخذنا بنظر الاعتبار طريقة ريس للجمع كاسلوب لتسوية حلول سلسلة فورييه لمشاكل الحرارة والموجه. عندما تم زيادة مرتبة متوسطات ريس, فان النتائج كانت تقاربية ولكن الحسابات العددية ازدادت. وعليه ولتقليل الحسابات العددية لحلول سلسلة فورييه المسواة فانا قمنا بالتحسين الامثل لتسوية الحلول المتعلقة بمشاكل اهتزاز اللوح ومشاكل التوصيل الحراري. وللتحسين الامثل لحلول

سلسلة فورييه المسواة تم استخدام اقل رتبة لمتوسطات ريس. كان اقل رتبة مستخدمة $s > (N-1/2 -l)$ بعد التحسين الامثل تم استخدام برنامج الحساب العددي Matlab للحلول العددية. هنا وجدنا التحسين الامثل لتسوية حلول السلاسل في نقطة ثابتة للوح عند الوقت الابتدائي و مؤشر حرج. تم الحصول على تقارب جيد بعد النقطة الحرجة.

APPROVAL PAGE

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DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

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Dedicated to...

My respected beloved parents Shohidul Islam and Fanshe Islam
My respected beloved Husband Dr. Arifutzzaman Rahat
May Allah accept our promises to which we begin with.
May Allah accept us here and here after.

ACKNOWLEDGEMENTS

First of all, my gratitude goes to Allah (swt), for His blessing to give me the ability and knowledge to complete the research work successfully. I am pleased to mention my supervisor Professor Dr Abdumalik Rakhimov for guidance and physical assistance to continue and complete my task. I must mention his consideration to take me as one of his PhD student under his supervision and to give his valuable time whenever I need. My profound gratitude also goes to my respective co-supervisor Professor Dr Ahsan Ali Khan for his wide help form the starting till finalization in terms of all kinds of support during my PhD tenure.

I would like to thank my mother Fanshe Islam for her support. Besides I would like to thank my uncle for his support at the initial time of my PhD.

My gratitude also goes out to my all respective specially my brother in law Lokman Hossian who have been urging me to complete this thesis. I sincerely would like to thank for his all of the support from the beginning to end of my PhD.

In special, honestly I would like acknowledge my husband Dr. Arifutzzaman Rahat's contribution to make my PhD in reality. Due to his long visionary thinking and respect to the education he allowed me to enroll in PhD. I realized his dedication and passion to start and continuing my PhD here in abroad. His input in my PhD expands from finding supervisor to managing all kinds' of financial matters. During his PhD I saw how much struggle he did, in spite of that he dragged my PhD study with those adverse condition in his life. He suffered relentlessly from all aspects to make my thesis in reality. Without his cordial desire and passionate input I would not able to finish my PhD thesis. It seems to me credit goes to him about a half to this outcome in my life for his cordial contributions. Thanking him would be very negligible in this regards.

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LIST OF SYMBOLS

ω	A family of bounded sets
C_n	Arbitrary complex number
$\theta(x, y, \lambda)$	Borel function
$\partial\Omega$	Boundary
$\chi_n(x)$	Characteristic function
A_0	Class of all homogeneous elliptic polynomials
C_ς	Constant
A_{nm}	Constant coefficient
B_{nm}	Constant coefficient
M_n	Constant coefficient
U_{n_m}	Constant coefficient
V_{n_m}	Constant coefficient
φ	Continious functions
$F_\lambda^j(A(\xi))$	Contribution of the stationary point $\xi_j(\omega)$
δ	Dirac delta function
$u_n(x)$	Eigenfunction
$A(\xi)$	Elliptic Polynomial
n_ξ	Exterior normal at the point ξ
Π	Family of bounded sets
f_n	Fourier coefficient
$f(x)$	Function
$C^l(\Omega)$	Holder class

x_0	Interior point
Δ	Laplace operator
$\lvert\alpha\rvert$	Length of multi-index
E	Linear topological space
ρ	Mass density
$H_p^l(R^N)$	Nikol'skii class
$\lVert . \rVert$	Norm
∂^γ	Partial derivative
ξ_i	Real number
x_i	Real numbers
E_λ^α	Riesz means of order α
$\hat{A}$	Self-adjoint extension
$P_{k,\gamma}(\phi)$	Semi norm
$E_\lambda f$	Spectral expansions of f
$\Theta(x,y,\lambda)$	Spectral function
U_r	Sphere of radious r centered on the origin
$U(x_0;r)$	Sphere of radious r centred on the x_0
$C_0^\infty(R^n)$	Subset of $C^\infty(R^n)$
$C^\infty(R^n)$	The space of all infinitely differentiable functions on R^n
χ_j	Truncating function
ψ	Wavelet
(x, y)	Position
∂^α	Partial Derivative

A_r	Narrower classes
$D_\lambda^s(x)$	Regularised Direchlet kernel
$\hat{f}(y)$	Fourier transformation
$L_1(T^N)$	Space of absolutely integrable function on T^N
$P_{K,\gamma}$	Semi-norms
$R_\lambda^s f(x)$	Bochner-Riesz means of Fourier Integral
$W_p^l(R^n)$	Sobolev classes
x_j	Variables
$\sigma_\lambda^s f(x)$	Riesz means of Fourier Series
$D_\lambda(x)$	Direchlet kernel
E'	Set of all Linear topological space
$L_p^\gamma(R^n)$	Liouville class
p'	Conjugate exponent of p
$\mathcal{J}'(R^N)$	Tempered distribution
$\mathfrak{D}'(R^n)$	The set of all distributions
$\Theta_\lambda^s(x)$	Bochner-Riesz means of the Fourier integral
$\hat{\Theta}_\lambda^s(\xi)$	Fourier transformation of the Bochner- Riesz kernel
T^N	*N* - dimensional torus
Ω_0	Proper sub domain
Z^N	Set of all vectors with integer components
2π	Period
a	Characteristic coefficient
A (n)	Elliptic Polynomial of degree 2(m + 1)
A(D)	Arbitrary differential operator
C	Continuous function

$\mathbb{C}$	Complex valued functional
$\mathcal{E}$	Local convex topological space
$\mathcal{E}'$	Conjugate space
h	Heat conductivity coefficient
I	Arbitrary interval
K	Compact subset
l	Non-negative integer
m	Positive integer number
nx	Inner product
r	Positive number
R	Real valued functional
R^N	Whole space
s	Order
T	Tension
t	Time
T^N	*N* - dimensional torus
u(x, y, t)	Deflection of membrane from equilibrium at position (x, y) and time t.
v(x, y, t)	Temperature of the plate at position (x, y) and time t
α, γ	Multi-index
λ	parameter
λ_k	Eigenvalues
Ω	Domain
$\mathcal{J}(R^n)$	The set of test functions
$\mathfrak{D}(R^n)$	The set of test functions

ε	Very small number
ω	Sequence of bounded set
$\boldsymbol{\varepsilon}'(R^n)$	The set of linear continuous functional on $\boldsymbol{\varepsilon}(R^n)$

CHAPTER ONE

INTRODUCTION

1.1 SOME BASIC NOTATION

Let us denote by R^N an N-dimensional Euclidean space and denote its points by $x = (x_1, x_{2,} ......x_N)$ where x_i, $i = 1, 2, ...N$ are real numbers. Let *r* is a positive number. The set of points $x \in R^N$ satisfying the inequality $|x - x_0| < r$ is defined as open sphere of radious *r* with its centre at the point x_0. The open sphere is denoted by $U(x_0; r)$. By $U_r = U(0; r)$ denotes this sphere of radious *r* centred on the origin. The point x_0 is said to be an interior point of a set E if there is a sphere $U(x_0; r)$ contained in this set. If all the points of a set E are interior then it is called an open set. If any two points of a set E can be joined through an unbroken line lying in this set then the set is called connected. A connected open set is called domain. In this thesis we denote domain by Ω. If the point x_0 is not an interior point and $U(x_0; r) \cap \Omega \neq \phi$ for any r > 0 then it is called point of boundary. Set of all point of boundary is denoted by $\partial\Omega$. A connected open subset is called bounded domain.

1.1.1 $C^\infty(R^N)$, $C_0^\infty(R^N)$ - Classes

The N – dimensional vector $\alpha =$ $(\alpha_1, \alpha_2, ...\alpha_N)$ with non negative integer components is called multi index and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ is called length of the multiindex. We also use the notation $x^\alpha = (x_1{}^{\alpha_1}, x_2{}^{\alpha_2} ......x_N{}^{\alpha_N})$, where $x \in R^N$ and $D^\alpha = (D_1{}^{\alpha_1}, D_2{}^{\alpha_2} ......D_N{}^{\alpha_N})$, where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ (i is the imaginary unit).

Let $f(x)$ is a function defined in R^N. By $D^\alpha f(x)$ we denote following partial derivative of the function $f(x)$

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ....\partial x_n^{\alpha_n}}, \qquad D^0 f(x) = f(x)$$

A set of (complex) functions f which is continuous together with the derivatives $D^\alpha f(x)$ for all α, where $|\alpha| \le p\,(0 \le p < \infty$, p is a non negative integer number) forms a class of functions $C^p(R^N)$ in R^N. The space of all infinitely differentiable functions on R^N is denoted by the symbol $C^\infty(R^N)$.

Let $\varphi \in C(R^N)$. The closure of the set of those points for which $\varphi(x) \neq 0$ is defined as the support of the continuous function φ. It is denoted by supp φ. The function φ is said to have compact support if supp φ is bounded set. The subset of $C^\infty(R^N)$ which contains the functions $C^\infty(R^N)$ with compact support in R^N is denoted by $C_0^\infty(R^N)$.

1.1.2 Linear Topological Space

A set E is defined as a linear topological space if E is a linear space and also a topological space and the operations of addition and multiplication of elements of E by real (complex) numbers are continuous with respect to the topology in E such that

a. If $z_0 = x_0 + y_0$, then for any neighbourhood U at the point z_0, there are neighbourhoods *V* and *W* of the points x_0 and y_0 such that $x + y \in U$ whenever $x \in V, y \in W$;

b. If $\alpha_0 x_0 = y_0$, then for any neighbourhood *U* of the point y_0, there is a neighbourhood *V* of the point x_0 and for any number $\varepsilon > 0$ such that $\alpha x \in U$ whenever $x \in V, |\alpha - \alpha_0| < \varepsilon$.

A mapping $f : E \to R$ is called real valued functional. Instead of R if we take ℂ then it is called complex valued functional.

A functional *f* defined on a linear topological space E is called linear if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

complex.

for all $x, y \in E$ and for all numbers α, β.

Similarly, a functional *f* defined on a linear topological space E is called continuous at the point $x_0 \in E$ if for any $\varepsilon > 0$, there is a neighbourhood U of x_0 such that

$$|f(x) - f(x_0)| < \varepsilon$$

for all $x \in U$. The functional *f* is said to be continuous on E if it is continuous at every point $x_0 \in E$.

The linear topological space E is called the space of test elements. The set of all linear topological space is called the space of distributions. It is denoted by E'.

1.2 FUNCTIONAL SPACES

1.2.1 The Spaces of Test Functions

Let us consider the set of test functions $\mathfrak{D} = \mathfrak{D}(R^N)$ all the infinitely differentiable functions in R^N with compact support. The convergence in $\mathfrak{D}$ is defined as follow:

the sequence of functions $\varphi_1, \varphi_2, \ldots$ from $\mathfrak{D}$ converges to the function φ belonging to $\mathfrak{D}$ if

1. there is a number r > 0 such that supp $\varphi_k \subset U_r$, $k = 1,2,3,.....$
2. for each $\alpha = (\alpha_1, \alpha_{2,} ......\alpha_N)$,

$$D^\alpha \varphi_k(x) \overset{x \in R^n}{\Rightarrow} D^x \varphi(x), \qquad k \to \infty .$$

So, we shall write $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathfrak{D}$.

Evidently $\mathfrak{D}$ is a linear space. The operation of differentiation $D^\beta \varphi(x)$ is continuous from $\mathfrak{D}$ into $\mathfrak{D}$. Now, from the definition of convergence in $\mathfrak{D}$ it is clear that if $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathfrak{D}$, then $D^\beta \varphi_k \to D^\beta \varphi$ as $k \to \infty$ in $\mathfrak{D}$.

The set of test functions, the supports of which are contained in the given domain Ω, is denoted by $\mathfrak{D}(\Omega)$ and we can write

$$\mathfrak{D}(\Omega) \subset \mathfrak{D}(R^N) = \mathfrak{D}$$

The following example of a test function different from a zero function is called the “hat function”.

$$\omega_\varsigma(x)=\begin{cases} C_\varsigma \exp(-\dfrac{\varsigma^2}{\varsigma^2-|x|^2}), & |x|\le\varsigma \\ 0, & |x|>\varsigma \end{cases},$$

We shall choose a constant C_ς suct that

$$\int \omega_\varsigma(x)dx=1,$$

that is

$$C_\varsigma \int \exp\left(-\frac{\varsigma^2}{\varsigma^2-|x|^2}\right)dx=1.$$

Let us consider the set of test functions $\mathcal{J}=\mathcal{J}(R^N)$ belonging to the class $C^\infty(R^N)$ which decreases as $|x|\to\infty$, together with all their derivatives and faster than any power of $|x|^{-1}$. Therefore, the sequence of functions $\varphi_1,\varphi_2,\ldots$ which is belongs to $\mathcal{J}$ converges to the functions $\varphi \in \mathcal{J}$, $\varphi_k\to\varphi$ as $k\to\infty$ in $\mathcal{J}$, if

$$x^\beta D^\alpha \varphi_k(x) \overset{x\in R^N}{\rightrightarrows} x^\beta D^\alpha\, \varphi(x), \qquad k\to\infty \tag{1.1}$$

for all α and β and for the continuous function $\varphi(x)$ at the point x.

Obviously $\mathcal{J}$ is a linear space. Moreover, $\mathfrak{D}\subset\mathcal{J}$ and the convergence in $\mathcal{J}$ follows from the convergence in $\mathfrak{D}$.

If $\varphi_k\to\varphi$ as $k\to\infty$ in $\mathfrak{D}$ then since the supports of φ_k are bounded independently of k then the limiting result (1.1) is valid for all α and β, that's mean $\varphi_k\to\varphi$ as $k\to\infty$ in $\mathcal{J}$. Though, $\mathcal{J}$ does not coincide with $\mathfrak{D}$. As an example, the function $\mathcal{J}(x)=\exp(-|x|^2)$ which belongs to $\mathcal{J}$ but it does not belong to $\mathfrak{D}$.